

# Geometric Equivariant Extension of Sections in GW Theory I

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## 1 Introduction

Let  $\tilde{\mathcal{B}} =: \tilde{\mathcal{B}}_{k,p}$  be the space of stable  $L_k^p$ -maps  $f : S^2 \rightarrow M$  from the Riemann sphere to a compact almost complex manifold  $(M, J)$ . Consider the bundle  $\tilde{\mathcal{L}}_{k-1,p} \rightarrow \tilde{\mathcal{B}}_{k,p}$  with the fiber  $(\tilde{\mathcal{L}}_{k-1,p})_f = L_{k-1}^p(S^2, \wedge_J^{0,1}(f^*TM))$ . It is well-known that  $(\tilde{\mathcal{L}}_{k-1,p}, \tilde{\mathcal{B}}_{k,p})$  is a Banach bundle of class  $C^\infty$ . The reparametrization group  $G =: \mathbf{PSL}(2, \mathbf{C})$  acting on  $(\tilde{\mathcal{L}}_{k-1,p}, \tilde{\mathcal{B}}_{k,p})$  continuously with local slices  $S_f$  for any  $f \in \tilde{\mathcal{B}}_{k,p}$ . Since the  $G$ -action is only continuous, the coordinate transformations between these local slices  $S_f$  as well as the transition functions between the local bundles  $(\tilde{\mathcal{L}}_{k-1,p}|_{S_f}, S_f)$  are only continuous in general. Because of this well-known difficulty in lack of differentiability, to establish the analytic foundation of GW theory by the method of [L] it is crucial to have sufficiently many smooth sections of the local bundle  $\tilde{\mathcal{L}}_{k-1,p}|_{S_f} \rightarrow S_f$  that are still smooth viewed in any other slices  $S_h$ .

In [L], we have given two different methods to construct such sections starting from an element  $\xi \in C^\infty(S^2, \wedge^{0,1}(f^*TM)) \subset L_{k-1}^p(S^2, \wedge^{0,1}(f^*TM)) = (\tilde{\mathcal{L}}_{k-1,p})_f$  for a smooth map  $f$ . The purpose of this paper is to explain one of the constructions, the geometric  $G$ -equivariant extension  $\xi_{\mathcal{O}_{S_f}}$  of  $\xi$ , and to improve the results in [L] on this construction. Throughout this paper, we will assume that  $k - 2/p > 1$ . Note that under this condition the  $L_{k-1}^p$  functions on  $S^2$  form a Banach algebra. Set  $m_0 = [k - 2/p]$ .

First observe that for a fixed  $\phi \in G$ , the action on  $(\tilde{\mathcal{L}}_{k-1,p}, \tilde{\mathcal{B}}_{k,p})$  is a  $C^\infty$  automorphism. Hence for the purpose of this paper and its sequels, we can replace  $G$  by  $G_e$  and only consider the local  $G_e$ -action. Here  $G_e$  is a local

chart of  $G$  containing the identity  $e \in G$  chosen as follows. Let  $\widetilde{W}(f)$  be a local coordinate chart of  $\widetilde{\mathcal{B}}_{k,p}$  centered at a smooth stable map  $f$  such that the bundle  $\widetilde{\mathcal{L}}_{k-1,p}|_{\widetilde{W}(f)} \rightarrow \widetilde{W}(f)$  is trivialized. Let  $\Gamma_f$  be the isotropy group of  $f$ . Since  $\Gamma_f$  is finite, we can choose  $G_e$  such that  $G_e \cap \Gamma_f = \{e\}$ . Then the  $G_e$ -action on  $\widetilde{W}(f)$  is free for sufficiently small  $\widetilde{W}(f)$ . Let  $S_f \subset \widetilde{W}(f)$  be a local slice of the action. By shrinking  $\widetilde{W}(f)$  if it is necessary, we may assume further that the local  $G_e$ -orbit  $\mathcal{O}_{S_f}$  of  $S_f$  is equal to  $\widetilde{W}(f)$ .

We now recall the two constructions of the  $G$ -equivariant extensions of  $\xi$  in [L].

The first one is obtained as follows. Under the above given trivialization, the given element  $\xi$  in the central fiber can be considered as a "constant", hence smooth section over  $\widetilde{W}(f)$ . Denote its restriction to the local slice  $S_f$  by  $\xi_{S_f}$  as a smooth section of  $\mathcal{L}_{k-1,p}|_{S_f} \rightarrow S_f$ . Then using the local  $G_e$ -action, we obtain a  $G_e$ -equivariant section  $\xi_{\mathcal{O}_{S_f}}$  over  $\widetilde{W}(f) = \mathcal{O}_{S_f}$ . More specifically,  $\xi_{\mathcal{O}_{S_f}}$  is defined by  $\xi_{\mathcal{O}_{S_f}}(h) = T(h)^*(\xi_{S_f}(h \circ T(h)))$ . Here the map  $T$  is defined in the following proposition proved in next section.

**Proposition 1.1** *Given a smooth stable map  $f : S^2 \rightarrow M$ , let  $S_f$  be the local slice  $S_f^E$  or  $S_f^{L_2}$  defined by the evaluation map or by the  $L_2$  orthogonal complement to the orbit  $\mathcal{O}_f$  of  $f$ . Then there is a  $C^{m_0}$  or  $C^\infty$  map  $T : \widetilde{W}(f) \rightarrow G_e$  accordingly such that for any  $h \in \widetilde{W}(f)$ ,  $h \circ T(h) \in S_f$ . Here  $m_0 = [k - 2/p]$ .*

It was proved by a direct computation in [L] that  $\xi_{\mathcal{O}_{S_f}}$  is of class  $C^1$ . A more conceptual proof of this will be given in the forth coming paper [L?], in which we will also show that generically  $\xi_{\mathcal{O}_{S_f}}$  is exact of class  $C^1$ . In contrast, we will show in this paper that the extension  $\xi_{\mathcal{O}_{S_f}}$  by the second construction is of class  $C^{m_0}$  or  $C^\infty$ . In other words, it has the same degree of the smoothness as  $T$  has.

To describe the second construction, let  $\cup_{i=1}^l V_i = M$  be a fixed open covering of  $M$ . Consider a corresponding covering  $\cup_{i=1}^l D_i = S^2$  of  $S^2$  satisfying the condition that  $h(D_i) \subset V_i, i = 1, \dots, l$  for any  $h \in \widetilde{W}(f)$ . In particular, for  $(h, \phi) \in S_f \times G_e$ ,  $h \circ \phi(D_i) \subset V_i$ . Fix a partition of unit  $\{\alpha_i, i = 1, \dots, l\}$  on  $S^2$  subordinate to the covering  $\{D_i, i = 1, \dots, l\}$ . Let  $\beta_i, i = 1, \dots, l$  be the cut-off functions defined on  $V'_i$  such that  $\beta_i = 1$  on  $V_i$ , where  $V_i \subset \subset V'_i$ . Fix a smooth local  $\mathbf{C}$ -frame  $\mathbf{t}_i = \{t_{i1}, \dots, t_{im}\}$  of  $(TM, J)$  on  $V'_i$ . Then for any  $\xi \in C^\infty(S^2, \wedge^{0,1}(f^*TM))$  containing in the central fiber  $\widetilde{\mathcal{L}}_f$ ,

$$\xi = \Sigma_i \alpha_i \cdot \xi = \Sigma_{i,\nu} \gamma_i^\nu \cdot t_{i\nu} \circ f = \Sigma_{i,\nu} \gamma_i^\nu \cdot (\beta_i t_{i\nu} \circ f).$$

Here  $\gamma_i^\nu \in C^\infty(S^2, \wedge_{S^2}^{0,1})$  supported on  $D_i$  with  $\alpha_i \cdot \xi = \Sigma_\nu \gamma_i^\nu \cdot t_{i\nu} \circ f$ , and  $\beta_i t_{i\nu} \circ f \in C^\infty(S^2, f^*TM)$  supported on  $f^{-1}(V'_i)$ .

We require that the  $G$ -equivariant extension  $\xi_{\mathcal{O}_{S_f}}$  to be defined satisfying the condition that

$$\xi_{\mathcal{O}_{S_f}} = \Sigma_i (\alpha_i \cdot \xi)_{\mathcal{O}_{S_f}} = \Sigma_{i,\nu} (\gamma_i^\nu)_{\mathcal{O}_{S_f}} \cdot (\beta_i t_{i\nu} \circ f)_{\mathcal{O}_{S_f}}.$$

Thus we only need to define the  $G$ -equivariant extensions  $(\gamma)_{\mathcal{O}_{S_f}}$  for  $\gamma \in C^\infty(S^2, \wedge_{S^2}^{0,1})$  and  $(\beta_i t_{i\nu} \circ f)_{\mathcal{O}_{S_f}}$ .

To get these extensions, note that in above we have used the decomposition  $L_{k-1}^p(S^2, \wedge^{0,1}(f^*TM)) = L_{k-1}^p(S^2, \wedge^{0,1}) \otimes L_{k-1}^p(S^2, f^*TM)$ . Here we have used the fact that  $L_{k-1}^p(S^2, \wedge^{0,1})$  and  $L_{k-1}^p(S^2, f^*TM)$  are the modules over the complex Banach algebra  $L_{k-1}^p(S^2, \mathbf{C})$ , and tensor product is taken over  $L_{k-1}^p(S^2, \mathbf{C})$ . A family version of this isomorphism gives rise the isomorphism of the bundles  $\tilde{\mathcal{L}}_{k-1,p} \simeq \Omega_{k-1,p}^{0,1} \otimes \tilde{\mathcal{T}}_{k-1,p}$ . Here  $\Omega_{k-1,p}^{0,1} = \tilde{\mathcal{B}}_{k,p} \times L_{k-1}^p(S^2, \wedge^{0,1}) \rightarrow \tilde{\mathcal{B}}_{k,p}$  is the trivial bundle and  $\tilde{\mathcal{T}}_{k-1,p}$  is the tangent bundle  $T\mathcal{B}_{k-1,p}$  restricted to  $\mathcal{B}_{k,p}$  but with the "standard" bundle structure obtained by using the  $J$ -invariant parallel transport on  $TM$ . One can show (in next section) that this bundle structure is  $C^\infty$  equivalent to that of  $T\mathcal{B}_{k-1,p}$ . However, the equivalence is not with respect to the induced complex structure by  $J$  on  $T\mathcal{B}_{k-1,p}$ .

Using the trivialization of  $\Omega_{k-1,p}^{0,1}$  above, we get the constant extension  $\gamma_{S_f}$  as a section of the bundle  $\Omega_{k-1,p}^{0,1}|_{S_f} \rightarrow S_f$ . Then  $\gamma_{\mathcal{O}_{S_f}} : \widetilde{W}(f) \rightarrow \Omega_{k-1,p}^{0,1}|_{\widetilde{W}(f)} =: \widetilde{W}(f) \times L_{k-1}^p(S^2, \wedge^{0,1})$  is defined to be  $\gamma_{\mathcal{O}_{S_f}}(h) = T(h)^*(\gamma_{S_f}(h \circ T(h)))$ . Let  $[\gamma_{S_f}] : S_f \rightarrow L_{k-1}^p(S^2, \wedge^{0,1})$  and  $[\gamma_{\mathcal{O}_{S_f}}] : \widetilde{W}(f) \rightarrow L_{k-1}^p(S^2, \wedge^{0,1})$  be the corresponding maps under the above trivialization. Then  $[\gamma_{S_f}](h) = \gamma$  for  $h \in S_f$  and  $[\gamma_{\mathcal{O}_{S_f}}](h) = T(h)^*(\gamma)$  for  $h \in \widetilde{W}(f)$ .

Hence  $[\gamma_{\mathcal{O}_{S_f}}]$  is smooth, and  $[\gamma_{\mathcal{O}_{S_f}}]$  have the same degree of the smoothness as  $T$  has by the following lemma.

**Lemma 1.1** *For  $\gamma \in C^\infty(S^2, \wedge^{0,1})$ , the orbit map  $\Psi_\gamma : G \rightarrow L_{k-1}^p(S^2, \wedge^{0,1})$  defined by  $\Psi_\gamma(\phi) = (\phi)^*(\gamma)$  is of class  $C^\infty$ .*

The  $G$ -equivariant extension  $(\beta_i t_{i\nu})_{\mathcal{O}_{S_f}}$  of  $\beta_i t_{i\nu}$  is defined by  $(\beta_i t_{i\nu})_{\mathcal{O}_{S_f}}(h) =: (\beta_i)_{\mathcal{O}_{S_f}}(h) \cdot (t_{i\nu})_{\mathcal{O}_{S_f}}(h) = \beta_i \circ h \cdot t_{i\nu} \circ h$  for  $h \in \widetilde{W}(f)$ . It follows from the definition that  $(\beta_i t_{i\nu})_{\mathcal{O}_{S_f}}$  is  $G$ -equivariant. We need to show that it is smooth.

**Proposition 1.2** *Let  $\mathcal{R} = \widetilde{W}(f) \times L_{k-1}^p(S^2, \mathbf{R}) \rightarrow \widetilde{W}(f)$  be the trivial bundle. Given a smooth function  $\beta : M \rightarrow \mathcal{R}$ , the section  $X_\beta : \widetilde{W}(f) \rightarrow \mathcal{R}$  defined by  $X_\beta(h) = \beta \circ h$  is of class  $C^\infty$ .*

If we replace  $\mathbf{R}$  by  $\mathbf{C}$ , the same conclusion holds. This proves the smoothness of  $(\beta_i)_{\mathcal{O}_{S_f}}$ .

Denote the section  $(t_{i\nu})_{\mathcal{O}_{S_f}}$  by  $T_{i\nu}$  for short. Instead of just proving the smoothness of  $T_{i\nu}$ , we introduce a further sheaf theoretic localization of the trivialization  $\widetilde{T}_{k-1,p}|_{\widetilde{W}(f)}$  by considering the corresponding sheafification, that will facilitate the proofs of the main theorems in this paper and its sequels. For our purpose here, we will only describe the sections of the sheafification on open sets  $D$  containing in some  $D_i$  of the above covering  $\{D_i, i = 1, \dots, l\}$ . To explain this, note that the standard constructions of the coordinate charts for  $\widetilde{\mathcal{B}}_{k,p}$  and bundle structure of  $\widetilde{\mathcal{T}}_{k-1,p} \rightarrow \widetilde{\mathcal{B}}_{k,p}$  can be obtained as an application of the theory of section functor on the category of FVB (vector bundles with fiber bundle morphisms) developed by Palais in [P]. The functorial nature of the section functor gives rise to the desired sheafification. More concretely, for  $D \subset D_i$ , let  $\widetilde{W}(f; D)$  be the space of  $L_k^p$ -maps  $h : D \rightarrow V_i \subset M$  with  $\|(h - f)|_D\|_{k,p}$  less than the prescribed small  $\epsilon$  and  $\widetilde{\mathcal{T}}_{k-1,p}(f; D) \rightarrow \widetilde{W}(f; D)$  be the corresponding bundle defined by  $(\widetilde{\mathcal{T}}_{k-1,p}(f; D))_h = L_{k-1}^p(D, h^*TM)$  for  $h \in \widetilde{W}_{k,p}(f; D)$ . For our purpose here, it is sufficient to consider  $D = D_i, i = 1, \dots, l$ . Then we get the functorial system of the bundles  $\widetilde{\mathcal{T}}_{k-1,p}(f; D_i) \rightarrow \widetilde{W}(f; D_i)$  with respect to the covering. Each bundle  $\widetilde{\mathcal{T}}_{k-1,p}(f; D_i) \rightarrow \widetilde{W}(f; D_i)$ , however, is considered as a morphism between the two sheaves and the sections  $T_{i\nu}, \nu = 1, \dots, m$  before become not just the sections of the bundle  $\widetilde{\mathcal{T}}_{k-1,p}(f; D_i) \rightarrow \widetilde{W}(f; D_i)$  but also morphisms between the two sheaves. Moreover, these sections form a frame of the bundle so that they give rise to a trivialization of the bundle. Similarly, the usual process of the local trivialization for the bundle  $\widetilde{\mathcal{T}}_{k-1,p}$  using the induced parallel transport by the  $J$ -invariant connection on  $TM$  carries over here and produces the "standard" local trivialization for  $\widetilde{\mathcal{T}}_{k-1,p}(f; D_i) \rightarrow \widetilde{W}(f; D_i)$ .

**Note:** One of the reasons that we have brought out the sheaf theoretic aspect in the above discussion is that while sections like  $T_{i\nu}$  or the "constant" sections of the local bundle  $\widetilde{\mathcal{T}}_{k-1,p}(f; D_i) \rightarrow \widetilde{W}(f; D_i)$  are morphisms of the corresponding sheaves, the  $G$ -equivariant extensions used in the first construction in [L] mentioned before can not be interpreted as such morphisms.

**Theorem 1.1** *The two local trivializations of  $\widetilde{\mathcal{T}}_{k-1,p}(f; D_i) \rightarrow \widetilde{W}(f; D_i)$  are  $C^\infty$  equivalent.*

The smoothness of the section  $(\beta_i t_{i\nu} \circ f)_{\mathcal{O}_{S_f}}$  then follows from the above proposition and theorem. This proves the following main theorem of this paper.

**Theorem 1.2** *Given a smooth stable map  $f$  and a smooth section*

$\xi \in L_{k-1}^p(S^2, \wedge^{0,1}(f^*TM))$ , *its "geometric"  $G$ -equivariant extension  $\xi_{\mathcal{O}_{S_f}}$  above has the same degree of smoothness as  $T$  has. Hence it is either of class  $C_0^m$  or of class  $C^\infty$  accordingly.*

**Note:** The  $C_0^m$ -smoothness of  $\xi_{\mathcal{O}_{S_f}}$  was already proved in [L] for the local slice  $S_f = S_f^E$  obtained by evaluation maps. The proof of this is clarified in this paper.

This paper is organized as follows.

In Sec. 2, we give a proof of the theorem on the smoothness of the  $L_k^p$ -section functor,  $\Gamma_{k,p}$ , applying to a smooth but possibly non-linear bundle map. This theorem is part of the theory of Palais in [P] on the Banach space valued section functor on the category of FVB (vector bundles with fiber bundle morphisms). As applications of the theorem, we prove the  $C^\infty$  smoothness of the coordinate transformations of  $\mathcal{M}ap_{k,p}$  and smoothness of transition functions of the bundle  $\widetilde{\mathcal{L}}_{k-1,p} \rightarrow \mathcal{M}ap_{k,p}$  as well as several related results. The sheaf theoretic localization of the local trivialization of  $\widetilde{\mathcal{L}}_{k-1,p}|_{\widetilde{W}(f)} \rightarrow \widetilde{W}(f)$  mentioned above is also discussed in this section.

In Sec. 3, we give the proofs for the two versions of the main theorem.

In Sec. 4, we consider an embedding  $M \rightarrow \mathbf{R}^m$  and its induced embedding  $\mathcal{M}ap_{k,p}(S^2, M) = L_k^p(S^2, M) \rightarrow L_k^p(S^2, \mathbf{R}^m)$ . Then we construct a smooth section  $\Delta$  of an infinite dimensional "obstruction" bundle  $\mathcal{N} \rightarrow \mathcal{T}$ . Here  $\mathcal{T}$  is an tubular neighborhood of  $\mathcal{M}ap_{k,p}(S^2, M)$  in  $L_k^p(S^2, \mathbf{R}^m)$ . The mapping space  $\mathcal{M}ap_{k,p}(S^2, M)$  then is realized as the zero locus of the section  $\Delta$ . Using this construction, we outline a different approach to GW theory. The full details of this part will be treated in a separate paper.

In Sec. 5, we present simpler proofs for the smoothness of  $L_k^p$ -norm and related results.

## 2 Coordinate Charts and Local Trivializations

We start with some basic definitions in order to fix our notations.

Let  $(M, \omega)$  be a compact symplectic manifold with a symplectic form  $\omega$ . Fix an  $\omega$ -compatible almost complex structure  $J$ . Denote the induced Riemannian metric by  $g_J(-, J-)$ . Consider the Riemann sphere  $\Sigma = (S^2, i_0)$  with the standard complex structure  $i_0$  and round metric. Assume that all the geometric data above are of class  $C^\infty$ . Let  $\mathcal{M}ap =: \mathcal{M}ap_{k,p}$  be the space of  $L_k^p$ -maps from  $S^2$  to  $M$  and  $\tilde{\mathcal{L}} =: \tilde{\mathcal{L}}_{k-1,p} \rightarrow \mathcal{M}ap_{k,p}$  be the bundle defined by  $\tilde{\mathcal{L}}_f = L_{k-1}^p(S^2, \wedge_{i_0, J}^{0,1}(f^*(TM)))$  for  $f \in \mathcal{M}ap_{k,p}$ . We will denote  $\wedge_{i_0, J}^{0,1}(f^*(TM))$  by  $\wedge^{0,1}(f^*(TM))$  for short.

For the discussion of this section, we assume further that  $k - 2/p > 1$ . It follows that (a) the space  $L_{k-1}^p(S^2, \mathbf{R})$  of  $L_k^p$ -functions on  $S^2$  is a Banach algebra; (b) any element in  $\mathcal{M}ap_{k,p}$  is at least of class  $C^1$ . Set  $m_0 = [k - 2/p] \geq 1$ .

It is well known that  $\mathcal{M}ap$  is a smooth Banach manifold and  $\tilde{\mathcal{L}}_{k-1,p} \rightarrow \mathcal{M}ap_{k,p}$  is a smooth Banach bundle. There are two ways to construct the local trivializations of the bundle  $\tilde{\mathcal{L}}_{k-1,p} \rightarrow \mathcal{M}ap_{k,p}$ . The first is the standard one given in the original paper [G] of Gromov, by identifying the fibers using the induced  $J$ -invariant parallel transport. The other can be obtained by considering the tangent bundle  $T\mathcal{M}ap_{k-1,p}$  first, then the simple relation between  $T\mathcal{M}ap_{k-1,p}$  and  $\tilde{\mathcal{L}}_{k-1,p}$  gives rise the local trivialization for  $\tilde{\mathcal{L}}_{k-1,p}$ . The bundle structure on  $T\mathcal{M}ap_{k-1,p}$  is given by Floer in [F1]. The smoothness of the transition functions between the local trivializations of  $T\mathcal{M}ap_{k-1,p}$  in [F1] is a consequence of the smoothness of the coordinate transformations of  $\mathcal{M}ap_{k-1,p}$ . On the other hand, a complete proof of the smoothness of the transition functions between the local trivializations in [G] as well as the smoothness of the transition functions between the above two types of local trivializations for  $T\mathcal{M}ap_{k-1,p}$  are not presented in the literature of the GW-theory. In addition to these local trivializations, in this paper we will introduce a sheaf theoretic type of local trivialization. As mentioned before, to prove the main theorem of this paper, it is crucial to show that all these local trivializations for  $T\mathcal{M}ap_{k-1,p}$  are  $C^\infty$ -equivalent. The main technique tool to prove such an equivalence as well as other results of similar nature in this paper is the work of Palais in [P] on the category of vector bundles with fiber bundle (hence, possibly non-linear )morphisms (FVB) and the Banach

space valued section functors on such a category. In fact, large part of the Floer's work in [F1] can be obtained as an application of the work in [P].

The key ingredient of the construction in [P] and [F1] is the following theorem that will be used repeatedly in this paper .

**Theorem 2.1** ([F1], [P]) *Given a compact smooth manifold  $M$ , let  $B_i \rightarrow M, i = 1, 2$  be the bundle of unit ball of the smooth vector bundle  $E_i \rightarrow M$ , and  $f : B_1 \rightarrow B_2$  be a possibly nonlinear smooth bundle map. Assume that  $k - \dim(M)/p > 0$  and that the space  $L_k^p(M)$  of  $L_k^p$ -functions on  $M$  is a Banach algebra. Then the induced map on  $L_k^p$ -section,  $f_* = \Gamma_{k,p}(f) : L_k^p(M, B_1) \rightarrow L_k^p(M, B_2)$  is of class  $C^\infty$ .*

*In fact, let  $df : B_1 \rightarrow L(E_1, E_2)$  be the fiberwise derivative of  $f$ , where  $L(E_1, E_2) \rightarrow M$  is the vector bundle with the fiber  $L(E_1, E_2)_m = L((E_1)_m, (E_2)_m)$  for  $m \in M$ . Then  $df$  is a smooth bundle map and*

$$D(f_*) = D(\Gamma_{k,p}(f)) = \Gamma_{k,p}(df) : \Gamma_{k,p}(B_1) = L_k^p(M, B_1) \rightarrow \Gamma_{k,p}(L(E_1, E_2)) =$$

$$L_k^p(M, L(E_1, E_2)) \subset L(\Gamma_{k,p}(E_1), \Gamma_{k,p}(E_2)) = L(L_k^p(M, E_1), L_k^p(M, E_2)).$$

**Proof:**

We only outline a proof. For more details, see [F1] and [P]. Clearly we only need to show that (1)  $f_* = \Gamma_{k,p}(f)$  is continuous and (2)  $D(f_*) = D(\Gamma_{k,p}(f))$  is equal to  $\Gamma_{k,p}(df)$ . Indeed, since the derivative  $df$  along the fiber is a smooth bundle map so that it plays the same role as  $f$  does, (1) and (2) imply that  $f_*$  is of class  $C^\infty$  by induction.

• Proof of (1):

Let  $s_1, s_2 \in L_k^p(M, B_1)$ . Then for  $j \leq k$ ,

$$\begin{aligned} \|D^j(f_*(s_1) - f_*(s_2))\|_{0,p} &= \|D^j(f \circ s_1 - f \circ s_2)\|_{0,p} \\ &= \int_0^1 \|D^j\left(\frac{df}{dt}(s_2 + t(s_1 - s_2))\right)\|_{0,p} dt \end{aligned}$$

is bounded by terms  $\|df\|_{C^k} \cdot \|D^i(s_1 - s_2)\|_{0,p}$  with  $i \leq j$ . In fact, a better bound is  $\|df\|_{k,q} \cdot \|D^i(s_1 - s_2)\|_{k,p}$ . Now for  $p \geq 2, q \leq 2$ ,  $L_k^p(S^2)$  is continuously embedded into  $L_k^q(S^2)$  so that we have the bound  $\|df\|_{k,p} \cdot \|D^i(s_1 - s_2)\|_{k,p}$ .

• Proof of (2):

For  $s \in L_k^p(M, B_1)$ ,  $\xi \in L_k^p(M, E_1)$  and  $j \leq k$ ,

$$\|D^j(f_*(s + t\xi) - f_*(s) - \Gamma_{k,p}(df)_s(t\xi))\|_{0,p} = \|D^j(f \circ (s + t\xi) - f \circ s - tdf_s \circ \xi)\|_{0,p}$$

$$\begin{aligned}
&= \|D^j(\int_0^1 (df(s+\nu t\xi)-df(s))d\nu(t\xi))\|_{0,p} \leq \int_0^1 \|D^j((df(s+\nu t\xi)-df(s))(t\xi))\|_{0,p}d\nu \\
&\leq \int_0^1 \int_0^1 \|D^j((d^2f(s+\mu\nu t\xi)(\nu t\xi))(t\xi))\|_{0,p}d\nu d\mu,
\end{aligned}$$

which is bounded by  $\|df^2\|_{C^k} \cdot \|t\xi\|_{k,p}^2$  under the assumption that  $L_k^p(M)$  is a Banach algebra. As remarked above, a better bound is  $\|df^2\|_{k,p} \cdot \|t\xi\|_{k,p}^2$   $\square$

**Note:** Using the better bounds in the proof above, we only to assume that (a) the bundle  $E_i \rightarrow M, i = 1, 2$  and the bundle map  $f : B_1 \rightarrow B_2$  are of class  $L_k^p$  (or  $C^k$ ); (b)  $f$  is of class  $C^\infty$  along the fiber such that the derivatives  $df$  and  $d^2f$  along the fiber as the corresponding bundle maps are of class  $L_k^p$  (or  $C^k$ ). Then the above theorem is still true.

We now apply above theorem to construct the standard coordinate transformations for  $\mathcal{Map}$  and the transition functions for  $\widetilde{\mathcal{L}}$ . In the following unless specified otherwise, we will assume that the center  $f$  of each coordinate is of class  $C^\infty$ . However, one can verify that the conditions in the above note are satisfied for the discussion below so that the same results are true when the center  $f \in \mathcal{Map}_{k,p}$  is of class  $L_k^p$ .

- • Smoothness of the coordinate transformations of  $\mathcal{Map}_{k,p}$ .

Recall the definition of natural coordinate chart  $\widetilde{W}(f)$  for  $f \in \mathcal{Map}_{k,p}$ . Consider the smooth bundle  $E = f^*(TM) \rightarrow S^2$ . Denote its sub-bundle of unit balls by  $B \rightarrow \Sigma = S^2$ . Let  $\hat{W} = \hat{W}_\epsilon$  be the  $\epsilon$ -ball of  $L_k^p(S^2, B) \subset L_k^p(S^2, f^*(TM))$ . Then the  $\epsilon$ -neighborhood of  $f$  in  $\mathcal{Map}$  is defined to be  $\widetilde{W}(f) =: \text{Exp}_f(\hat{W})$ , where  $(\text{Exp}_f\xi)(x) = \exp_{f(x)}\xi(x)$ .

Given two smooth elements  $f_i, i = 1, 2 \in \mathcal{Map}$ , the above theorem implies that the coordinated transformation  $\Psi_{21} : \widetilde{W}_{f_1} \rightarrow \widetilde{W}_{f_2}$  is of class  $C^\infty$ . Indeed, assume  $f_1$  and  $f_2$  are  $C^0$ -close each other so that the map  $\psi_{21} : B_1 \rightarrow B_2$  given by  $\psi_{21}(b) = \exp_{f_2(x)}^{-1} \circ \exp_{f_1(x)}(b)$  is well-defined on  $B_1$ , where  $b \in (B_1)_x$ . Then  $\psi_{21}$  is of class  $C^\infty$ , and the coordinated transformation  $\Psi_{21} : \widetilde{W}_{f_1} \rightarrow \widetilde{W}_{f_2}$  is the restriction of  $(\psi_{21})_* = \Gamma_{k,p}(\psi_{21})$  to  $\widetilde{W}(f_1)$ . Hence by the above theorem  $\Psi_{21}$  is of class of  $C^\infty$ .

- • Smoothness of the transition functions of  $\widetilde{\mathcal{L}}$ .

Next recall the local trivialization of the bundle  $\widetilde{\mathcal{L}} = \widetilde{\mathcal{L}}_{k-1,p}$  over the local chart  $\widetilde{W}_f$ . For any  $h \in \widetilde{W}_f$  and  $\xi \in \widetilde{\mathcal{L}}_f$ , the local trivialization  $\Pi_f : \widetilde{W}_f \times \widetilde{\mathcal{L}}_f \rightarrow \widetilde{\mathcal{L}}|_{\widetilde{W}_f}$  is given by  $\Pi_f(h, \xi)(x) = P_{h(x)f(x)}\xi(x)$  for  $x \in S^2$  where



$P_{h(x)f(x)}$  is the induced action by the  $J$ -invariant parallel transport along the shortest geodesic from  $f(x)$  to  $h(x)$ .

Given  $f_1$  and  $f_2$  in  $\mathcal{Map}$  as above, the transition function between two local trivializations then is defined by  $\Pi_{21} =: \Pi_{f_2}^{-1} \circ \Pi_{f_1} : \widetilde{W}_{f_1} \times \widetilde{\mathcal{L}}_{f_1} \rightarrow \widetilde{W}_{f_2} \times \widetilde{\mathcal{L}}_{f_2}$ . To see that  $\Pi_{21}$  is of class  $C^\infty$ , we make some reductions.

Note that any  $\xi \in \widetilde{\mathcal{L}}_f$  is a summation of the elements  $\eta \otimes \gamma$  with  $\eta \in L_{k-1}^p(S^2, f^*TM)$  and  $\gamma \in C^\infty(S^2, \wedge^{0,1})$ . In other words,  $\widetilde{\mathcal{L}}_f = L_{k-1}^p(S^2, f^*TM) = C^\infty(S^2, \wedge^{0,1}) \otimes L_{k-1}^p(S^2, \wedge^{0,1}(f^*TM))$ . Note that identification of the fibers induced by the parallel transport in local trivialization for  $\widetilde{\mathcal{L}}$  only acts on  $L_{k-1}^p(S^2, f^*TM)$ . This implies that  $\widetilde{\mathcal{L}} \simeq \Omega^{0,1} \otimes \widetilde{\mathcal{T}}_{k-1,p}$ . Here  $\Omega^{0,1} \rightarrow \mathcal{B}$  is the trivial bundle defined by  $\Omega_f^{0,1} = C^\infty(S^2, \wedge^{0,1})$ , and the bundle  $\widetilde{\mathcal{T}}_{k-1,p}$  with the fiber  $\widetilde{\mathcal{T}}_{k-1,p}|_f = L_{k-1}^p(S^2, f^*TM)$  has the local trivialization induced by the parallel transport as for  $\widetilde{\mathcal{L}}$ . Thus we only need to show that the corresponding transition function  $\Pi_{21}$  for  $\widetilde{\mathcal{T}}_{k-1,p}$  is of class  $C^\infty$ .

To this end, let  $p_i : B_i \subset E_i \rightarrow S^2$  be the unit ball bundle of  $E_i =: f_i^*TM$  before. Consider the pull-back bundles  $\mathbf{E}_i = (Exp_{f_i})^*(TM) \rightarrow B_i, i = 1, 2$  by the maps  $Exp_{f_i} : B_i \subset E_i \rightarrow M$ . Then the fiber  $(\mathbf{E}_i)_b$  at  $b \in B_i$  with  $p_i(b) = x$  is  $T_{exp_{f_i(x)}b}M$ .

Next consider the pull-back of the bundles  $E_i \rightarrow S^2$  by  $p_i : B_i \rightarrow S^2$ ,  $p_i^*(E_i) \rightarrow B_i$ . The bundle map  $\pi_{f_i} : p_i^*(E_i) \rightarrow \mathbf{E}_i$  is defined by  $\pi_{f_i}(b, e) = (b, P(exp_{f_i(x)}b, f_i(x))(e))$  for  $(b, e) \in p_i^*(E_i)$  with  $b \in B_i, p_i(b) = x$  and  $e \in (E_i)_x$ , where  $P(exp_{f_i(x)}b, f_i(x))(e)$  is the parallel transport of  $e$  along the geodesic from  $f_i(x)$  to  $exp_{f_i(x)}b$  by a fixed  $J$ -invariant connection on  $(TM, J)$ . It is easy to see that  $\pi_{f_i}$  is of class  $C^\infty$ . Then the bundle map  $\pi_{21} =: \pi_{f_2}^{-1} \circ \pi_{f_1} : p_1^*(E_1) \rightarrow p_2^*(E_2)$  is of class  $C^\infty$  as well.

In order to applying the above theorem, we need to find the corresponding maps between the relevant vector bundles on  $S^2$  rather than on  $B_i$  as above. To this end, consider the bundle  $H = H_{21} = Hom(E_1, E_2) \rightarrow S^2$  and the bundle map  $T : B_1 \rightarrow H$  given by  $T(b)(e) = \pi_{21}(b, (p_1^*(e))(b))$ . Here  $p_1^*(e)$  is the "vertical" lifting of  $e \in (E_1)_x$ , which is a section of  $p_1^*(E_1)$  along the fiber  $(B_1)_x$  and  $(p_1^*(e))(b)$  is its value at  $b \in (B_1)_x$ . Then the smoothness of  $\pi_{21}$  implies that  $T$  is of class  $C^\infty$ .

Now applying the  $L_{k-1}^p$ -section functor  $\Gamma_{k-1,p}$ , we get Banach spaces  $\Gamma_{k-1,p}(H)$  and  $\Gamma_{k-1,p}(E_i), i = 1, 2$ . The pairing  $\langle -, - \rangle : \Gamma_{k-1,p}(H) \times \Gamma_{k-1,p}(E_1) \rightarrow \Gamma_{k-1,p}(E_2)$  is of class  $C^\infty$ . By the above theorem, the smoothness of  $T$  implies that  $\Gamma_{k-1,p}(T) : \Gamma_{k,p}(B_1) \rightarrow \Gamma_{k,p}(H) \rightarrow \Gamma_{k-1,p}(H)$  is of class  $C^\infty$ .

Consequently the map  $\mathbf{T} =: \langle -, - \rangle \circ (\Gamma_{k-1,p}(T) \times Id) : \Gamma_{k,p}(B_1) \times$

$\Gamma_{k-1,p}(E_1) \rightarrow \Gamma_{k-1,p}(H) \times \Gamma_{k-1,p}(E_1) \rightarrow \Gamma_{k-1,p}(E_2)$  is of class  $C^\infty$ .

Now  $\Pi_{21}(\xi, \eta) = (\Psi_{21}(\xi), \mathbf{T}(\xi, \eta))$ . This implies that  $\Pi_{21}$  is of class  $C^\infty$ .

•• Smooth equivalence between the local trivializations of  $T\tilde{\mathcal{B}}_{k-1,p}$  and  $\tilde{\mathcal{T}}_{k-1,p}$

Note that if we replace  $k-1$  by  $k$ , the bundle  $\tilde{\mathcal{T}}_{k,p}$  is just the  $(C^\infty)$  tangent bundle  $T\tilde{\mathcal{B}}_{k,p}$  but with different local trivializations. The local trivializations for  $T\tilde{\mathcal{B}}_{k,p}$  induce the local trivializations and hence a  $C^\infty$  bundle structure for  $T\tilde{\mathcal{B}}_{k-1,p}$ . We now show that the set theoretic identification  $T\tilde{\mathcal{B}}_{k-1,p} \rightarrow \tilde{\mathcal{T}}_{k-1,p}$  is in fact a bundle isomorphism of class  $C^\infty$ . We will prove this by showing that the transition functions between of the local trivializations of  $T\tilde{\mathcal{B}}_{k-1,p}$  and  $\tilde{\mathcal{T}}_{k-1,p}$  are of class  $C^\infty$ .

First recall the local trivialization of  $T\tilde{\mathcal{B}}_{k-1,p} \rightarrow \tilde{\mathcal{B}}_{k,p}$  over a local chart  $\tilde{W}(f)$ ,  $D_f =: D(\text{Exp}_f) : \tilde{W}(f) \times (T\tilde{\mathcal{B}})_f \rightarrow T\tilde{\mathcal{B}}|_{\tilde{W}_f}$  is given by  $D_f(h, \xi)(x) = (D(\text{Exp}_{f(x)}))_{h(x)}(\xi(x))$  for  $x \in S^2$ . To show that the transition function between  $D_f$  and  $\Pi_f$  is of class  $C^\infty$ , we need to find the corresponding bundle maps as we did for the transition function  $\Pi_{21}$ .

Consider the bundle  $p : E = f^*(TM) \rightarrow S^2$ , its unit ball bundle  $p : B \rightarrow S^2$ , the pull-back bundles  $p^*(E) \rightarrow B$  by  $p$  and  $\mathbf{E} =: (\text{Exp}_f)^*(TM) \rightarrow B$  by  $\text{Exp}_f : B \subset E \rightarrow M$ . Define the bundle isomorphism  $I = I_f : p^*(E) \rightarrow \mathbf{E}$  by  $I(b, e) = (b, (D(\text{Exp}_{f(x)}))_{\text{Exp}_f^{-1}(b)}(e))$  where  $x = p(b)$ . Then  $I$  is of class  $C^\infty$ . Let  $A = A_f =: \pi_f^{-1} \circ I_f : p^*(E) \rightarrow p^*(E)$  be the bundle automorphism. Then  $A$  is of class  $C^\infty$ . As before define a  $C^\infty$  bundle map  $T : B \rightarrow H$  by  $T(b)(e) = A(b, (p^*(e))(b))$ , where the bundle  $H = \text{Hom}(E, E) \rightarrow S^2$ . Here  $p^*(e)$  is the section the fiber  $(B_1)_x$  of the vertical lifting of  $e \in (E)_x$  and  $(p^*(e))(b)$  is the its value at  $b \in (B_1)_x$ . Then the argument before implies that  $\Gamma_{k-1,p}(T) : \Gamma_{k,p}(B) \rightarrow \Gamma_{k,p}(H) \rightarrow \Gamma_{k-1,p}(H)$  is of class  $C^\infty$  so that the map  $\mathbf{T} =: \langle -, - \rangle \circ (\Gamma_{k,p}(T) \times Id) : \Gamma_{k,p}(B) \times \Gamma_{k-1,p}(E) \rightarrow \Gamma_{k-1,p}(H) \times \Gamma_{k-1,p}(E) \rightarrow \Gamma_{k-1,p}(E)$  is of class  $C^\infty$ .

Now the transition function  $\mathbf{A}$  between the two local trivializations  $D_f$  and  $\Pi_f$  above is given by  $\mathbf{A}(\xi, \eta) = (\xi, \mathbf{T}(\xi, \eta))$ . This implies that  $\mathbf{A}$  is of class  $C^\infty$ , hence proves the following proposition.

**Proposition 2.1** *The transition functions between the local trivializations of  $\tilde{\mathcal{T}}_{k-1,p} \simeq T\tilde{\mathcal{B}}_{k-1,p}$  are of class  $C^\infty$ .*

**Corollary 2.1** *The section  $s =: \bar{\partial}_J : \tilde{\mathcal{B}}_{k,p} \rightarrow: \tilde{\mathcal{L}}_{k-1,p}$  defined by  $s(f) = df + J(f) \cdot df \circ i_0$  is of class  $C^\infty$ .*

**Proof:**

The result is proved in [F] and [P] for  $s : \tilde{\mathcal{B}}_{k,p} \rightarrow \tilde{\mathcal{L}}_{k-1,p}$  with respect to the smooth structure on  $\tilde{\mathcal{L}}_{k-1,p}$  induced from  $T\tilde{\mathcal{B}}_{k-1,p}$ .  $\square$

• Further sheaf-theoretic localization of a trivialization of the bundle  $\tilde{\mathcal{T}}_{k-1,p}$

To facilitate the proof of the main theorem, we now introduce a sheaf-theoretic localization of certain sections of the local bundle  $\tilde{\mathcal{T}}_{k-1,p}|_{\widetilde{W}(f)} \rightarrow \widetilde{W}(f)$ .

Let  $\cup_{i=1}^l D_i = S^2$  be an fixed open covering of  $S^2$  and  $\alpha_i, i = 1, \dots, l$  is a partition of unit subordinated to the covering. Each  $\alpha_i$  induces a bundle morphism  $E_{\alpha_i} : \tilde{\mathcal{T}}_{k-1,p} \rightarrow \tilde{\mathcal{T}}_{k-1,p}$  defined by  $E_{\alpha_i}(\eta) = \alpha_i \cdot \eta$  for  $\eta \in \tilde{\mathcal{T}}_{k-1,p}|_h = L_{k-1}^p(S^2, h^*TM)$  with  $h \in \tilde{\mathcal{B}}_{k,p}$ . It is easy to see that  $E_{\alpha_i}$  is smooth. Indeed in the any of above local trivialization  $\tilde{\mathcal{T}}_{k-1,p}|_{\widetilde{W}(f)} \simeq \widetilde{W}(f) \times L_{k-1}^p(S^2, f^*TM)$ , the corresponding map  $[E_{\alpha_i}]_f : \widetilde{W}(f) \times L_{k-1}^p(S^2, f^*TM) \rightarrow \widetilde{W}(f) \times L_{k-1}^p(S^2, f^*TM)$  given by  $[E_{\alpha_i}]_f(h, \xi) = (h, \alpha_i \cdot \xi)$  is "linear" and continuous, hence smooth.

Thus given a section  $s : \tilde{\mathcal{B}}_{k,p} \rightarrow \tilde{\mathcal{T}}_{k-1,p}$ ,  $s$  is of class  $C^k, k = 0, 1, \dots, \infty$ , if and only if  $E_{\alpha_i}(s)$  is for all  $i = 1, \dots, l$ .

Now let  $\cup_{i=1}^l V_i = M$  be a fixed open covering. Assume that the cover  $D$  of  $S^2$  above satisfies the condition that  $h(D_i) \subset V_i, i = 1, \dots, l$  for any  $h \in \widetilde{W}(f)$ . In particular, for  $(h, \phi) \in S_f \times G_e$ ,  $h \circ \phi(D_i) \subset V_i$ . Let  $V_i \subset \subset V'_i, i = 1, \dots, l$  and  $\beta_i, i = 1, \dots, l$  be cut-off functions defined on  $V'_i$  such that  $\beta_i = 1$  on  $V_i$ . For each  $i$ , let  $\alpha'_i$  be a cut-off function on  $S^2$  supported on  $D'_i$  with  $D_i \subset \subset D'_i$  and  $\alpha'_i = 1$  on  $D_i$ . Fix a smooth local frame  $\mathbf{t}_i = \{t_{i1}, \dots, t_{im}\}$  of  $TM$  on  $V'_i$ . Then for  $h \in \widetilde{W}(f)$  and  $x \in D_i$ ,

$$\begin{aligned} (E_{\alpha_i}(s))(h)(x) &= \Sigma_\nu(s_i^\nu(h))(x) \cdot t_{i\nu}(h(x)) = \Sigma_\nu(s_i^\nu(h))(x) \cdot \beta_i(h(x))t_{i\nu}(h(x)) \\ &= \Sigma_\nu(s_i^\nu(h))(x) \cdot \alpha'_i(x)t_{i\nu}(h(x)). \end{aligned}$$

Note that by our assumption for any  $h \in \widetilde{W}(f)$ , (1) the pull back frame  $h^*(\mathbf{t}_i)$  is defined over  $h^{-1}(V_i)$  that contains  $D_i$  hence the support of  $(E_{\alpha_i}(s))(h)$ ; (2) the pull back of the global sections sections  $\beta_i \mathbf{t}_i =: \{\beta_i t_{i1}, \dots, \beta_i t_{im}\}$ ,  $h^*(\beta_i \mathbf{t}_i)$  are global sections of the bundle  $h^*TM \rightarrow S^2$ , which plays the role of a "global frame" of the bundle  $h^*TM \rightarrow S^2$  to express the sections like  $(E_{\alpha_i}(s))(h)$  supported on  $D_i$ ; (3) by multiplying by  $\alpha'_i$  the frame  $h^*(\mathbf{t}_i)$  above

becomes  $E_{\alpha'_i}(h^*(\mathbf{t}_i)) =: \alpha'_i h^*(\mathbf{t}_i)$  that is also a "global frame" of the bundle  $h^*TM \rightarrow S^2$ .

Hence for each  $i$ , we get three corresponding collections of sections, denoted by  $\mathbf{T}_i = \{T_{i1}, \dots, T_{im}\}$ ,  $E_{\beta_i}(\mathbf{T}_i) = \{E_{\beta_i}(T_{i1}), \dots, E_{\beta_i}(T_{im})\}$  and  $E_{\alpha'_i}(\mathbf{T}_i) = \{E_{\alpha'_i}(T_{i1}), \dots, E_{\alpha'_i}(T_{im})\}$ . The section  $T_{i\nu} : \widetilde{W}(f; D_i) \rightarrow \widetilde{\mathcal{T}}_{k-1,p}(f; D_i)$  is defined by  $T_{i\nu}(h) = t_{ij} \circ h$  for  $h \in \widetilde{W}(f; D_i)$ . Here  $\widetilde{W}(f; D_i)$  consists of  $L_k^p$ -maps  $h : D_i \rightarrow V_i \subset M$  such that  $\|(h - f)|_{D_i}\|_{k,p}$  less than the prescribed small  $\epsilon$ ; and the bundle  $\widetilde{\mathcal{T}}_{k-1,p}(f; D_i)$  is defined by the same formula as before,  $(\widetilde{\mathcal{T}}_{k-1,p}(f; D_i))_h = L_{k-1}^p(D_i, h^*TM)$  for  $h \in \widetilde{\mathcal{T}}_{k-1,p}(f; D_i)$ . The section  $E_{\beta_i}(T_{i\nu}) : \widetilde{W}(f) \rightarrow \widetilde{\mathcal{T}}_{k-1,p}|_{\widetilde{W}(f)}$  is defined by  $E_{\beta_i}(T_{i\nu})(h) = (\beta_i \cdot t_{ij}) \circ h$  for  $h \in \widetilde{W}$ . The section  $E_{\alpha'_i}(T_{i\nu}) : \widetilde{W}(f) \rightarrow \widetilde{\mathcal{T}}_{k-1,p}|_{\widetilde{W}(f)}$  is defined similarly.

The main theorem is the following.

**Theorem 2.2** *The section  $E_{\alpha'_i}(T_{i\nu}) \& E_{\beta_i}(T_{i\nu}) : \widetilde{W}(f) \rightarrow \widetilde{\mathcal{T}}_{k-1,p}|_{\widetilde{W}(f)}$  as well as the section  $T_{i\nu} : \widetilde{W}(f; D_i) \rightarrow \widetilde{\mathcal{T}}_{k-1,p}(f; D_i)$  are smooth of class  $C^\infty$ .*

To simplify our notations, we will drop the subscript  $i$  for the discussion below. The theorem will be derived as a corollary of the smooth equivalence of the following two local trivializations of the bundle  $\widetilde{\mathcal{T}}_{k-1,p}(f; D) \rightarrow \widetilde{W}(f; D)$ . The first local trivialization is just the standard one induced by the  $J$ -invariant parallel transport. It can be reformulate as follows. Let  $Q_\nu : \widetilde{W}(f; D) \rightarrow \widetilde{\mathcal{T}}_{k-1,p}(f; D)$  be the corresponding section obtained as the "constant" extension of the point-section  $t_\nu \circ f|_D \in (\widetilde{\mathcal{T}}_{k-1,p}(f; D))_f = L_{k-1}^p(D, f|_D^*TM)$  by the standard local trivialization induced by the parallel transport along short geodesics. Then the local trivialization  $\Pi^1 : \widetilde{W}(f; D) \times L_{k-1}^p(S^2, f_D^*TM) \rightarrow \widetilde{\mathcal{T}}_{k-1,p}(f; D)$  is given by  $\Pi^1(h, \xi) = \Sigma_\nu \xi^\nu Q_\nu(h)$  for  $(h, \xi) \in \widetilde{W}(f; D) \times L_{k-1}^p(S^2, f_D^*TM)$  with  $\xi = \Sigma_\nu \xi^\nu Q_\nu(f|_D)$ , where  $\xi^\nu \in L_{k-1}^p(D, \mathbf{C})$ . Denote the collection of  $Q_\nu, \nu = 1, \dots, m$  by  $\mathbf{Q} (= \mathbf{Q}_i, i = 1, \dots, l)$ . Thus like  $\mathbf{T}$ ,  $\mathbf{Q}$  is also a frame for  $\widetilde{\mathcal{T}}_{k-1,p}(f; D)$ .

The second trivialization  $\Pi^2 : \widetilde{W}(f; D) \times L_{k-1}^p(S^2, f_D^*TM) \rightarrow \widetilde{\mathcal{T}}_{k-1,p}(f; D)$  is given similarly by replacing  $Q_\nu$  above by  $T_\nu$ . Hence  $\Pi^2(h, \xi) = \Sigma_\nu \xi^\nu T_\nu(h)$  for  $(h, \xi) \in \widetilde{W}(f; D) \times L_{k-1}^p(S^2, f_D^*TM)$  with  $\xi = \Sigma_\nu \xi^\nu T_\nu(f|_D)$ .

**Proposition 2.2** *The two local trivializations  $\Pi^1 \& \Pi^2 : \widetilde{W}(f; D) \times L_{k-1}^p(S^2, f_D^*TM) \rightarrow \widetilde{\mathcal{T}}_{k-1,p}(f; D)$  are smoothly equivalent.*

**Proof:**

As before, the proposition will be proved by applying the theory of TVB in [P] by finding the corresponding bundle maps. In steady of essentially repeating what we did before, we give an "universal" treatment that works for the case here as well as the cases before with necessary modifications.

Let  $V$  be one of those  $V_i, i = 1, \dots, l$  above. Denote the bundle  $TV$  by  $p : E =: TV \rightarrow S^2$ , its unit ball bundle by  $p : B \rightarrow S^2$ . Consider the pull-back bundles  $p^*(E) \rightarrow B$  and  $\mathbf{E} =: (Exp)^*(E) \rightarrow B$  by the maps  $p : B \rightarrow S^2$  and  $Exp = Exp_V : B \subset E \rightarrow M$  respectively.

Define the following two bundle isomorphisms  $I_1, I_2 : p^*(E) \rightarrow \mathbf{E}$  as follows. For any  $(b, e) \in p^*(E)$  with  $b \in B, p(b) = x$  and  $e \in E_x$ ,  $I_1 =: p^*(E) \rightarrow \mathbf{E}$  by  $I_1(b, e) = (b, P(exp_x b, x)(e))$ , where  $P(exp_x b, x)(e)$  is the parallel transport of  $e$  along the geodesic from  $x$  to  $exp_x b$  by the fixed  $J$ -invariant connection on  $(TM, J)$ . Then  $I_1$  is of class  $C^\infty$ . Thus this is the bundle isomorphism relevant to the standard trivialization induced from the parallel transport. The second bundle isomorphisms  $I_2 : p^*(E) \rightarrow \mathbf{E}$  is defined by  $I_2(b, e) = (b, \Sigma_\nu a^\nu t_\nu(exp_x b))$  for  $b \in B, p(b) = x$  and  $e = \Sigma_\nu a^\nu t_\nu(x) \in E_x$ . Again  $I_2$  is of class  $C^\infty$ .

Let  $A =: (I_2)^{-1} \circ I_1 : p^*(E) \rightarrow p^*E$  be the bundle automorphism. Then  $A$  is of class  $C^\infty$ .

Now let  $f_D = f|_D : D \rightarrow V \subset M$ . Consider the pull-back bundle  $p_f = p_{f_D} : E_f = f_D^*(E) \rightarrow D$ , its unit ball bundle  $p_f : B_f = f_D^*(B) \rightarrow D$  as well as the pull-backs of all the related commutative diagrams. In particular, consider the pull-back bundle  $p_f^*(E_f) \rightarrow B_f$  of  $E_f$  by the map  $p_f : B_f \rightarrow D$ . Then the bundle automorphism  $A$  induces a corresponding  $C^\infty$ -bundle automorphism  $A_f : p_f^*(E_f) \rightarrow p_f^*(E_f)$ .

As before define a  $C^\infty$  bundle map  $T_f : B_f \rightarrow H_f$  by  $T(b)(e) = A(b, (p^*(e))(b))$ , where the bundle  $H_f = Hom(E_f, E_f) \rightarrow D$ . Here  $p^*(e)$  is the vertical lifting of  $e \in (E_f)_x$  along the fiber  $(B_f)_x$  and  $(p_f^*(e))(b)$  is its value at  $b \in (B_f)_x$ . Then the argument before implies that  $\Gamma_{k-1,p}(T_f) : \Gamma_{k,p}(B_f) \rightarrow \Gamma_{k,p}(H_f) \rightarrow \Gamma_{k-1,p}(H_f)$  is of class  $C^\infty$  so that the map  $\mathbf{T}_f =: \langle -, - \rangle \circ (\Gamma_{k,p}(T_f)) \times Id : \Gamma_{k,p}(B_f) \times \Gamma_{k-1,p}(E_f) \rightarrow \Gamma_{k-1,p}(H_f) \times \Gamma_{k-1,p}(E_f) \rightarrow \Gamma_{k-1,p}(E_f)$  is of class  $C^\infty$ .

Now the transition function  $\mathbf{A}_f = \mathbf{A}_{f_D}$  between the two local trivializations  $\Pi_f^1$  and  $\Pi_f^2$  above is given by  $\mathbf{A}_f(\xi, \eta) = (\xi, \mathbf{T}(\xi, \eta))$ . This implies that  $\mathbf{A}_f$  is of class  $C^\infty$ , hence proves the proposition.  $\square$

•• Proof of the theorem:

Clearly the smoothness of  $T_\nu$  follows from the above proposition.

The section  $E_\alpha(T_\nu) : \widetilde{W}(f) \rightarrow \widetilde{\mathcal{T}}_{k-1,p}|_{\widetilde{W}(f)}$  considered as a map  $\widetilde{W}(f) \rightarrow$

$L_{k-1}^p(S^2, f^*TM) = \widetilde{\mathcal{T}}_{k-1,p}|_f$  is a composition  $E_\alpha(T_\nu) = E_\alpha \circ T_\nu \circ R : \widetilde{W}(f) \rightarrow \widetilde{W}(f; D) \rightarrow L_{k-1}^p(S^2, f_D^*TM) \rightarrow L_{k-1}^p(S^2, f^*TM)$ . Here  $R : \widetilde{W}(f) \rightarrow \widetilde{W}(f; D)$  is the restriction map and  $E_\alpha : L_{k-1}^p(S^2, f_D^*TM) \rightarrow L_{k-1}^p(S^2, f^*TM)$  is the multiplication by  $\alpha$  with support of  $\alpha$  containing in  $D$ . Clearly both  $R$  and  $E_\alpha$  are continuous and linear (in the local chart for  $R$ ), hence smooth so that  $E_\alpha(T_\nu)$  is a smooth section.

To see the smoothness of  $E_\beta(T_\nu) : \widetilde{W}(f) \rightarrow \widetilde{\mathcal{T}}_{k-1,p}|_{\widetilde{W}(f)}$ , recall that support of  $\beta$  is contained in  $V'$  such that  $\beta = 1$  on  $V \subset\subset V'$ . We will assume that the local frame  $\mathbf{t}$  is defined on slightly larger open neighborhood  $V^{(4)}$  of  $\bar{V}''$  with  $V' \subset\subset V''' \subset\subset V''$ . Then we may assume that  $f^{-1}(\bar{V}'') \neq S^2$ , otherwise the image of any  $h \in \widetilde{W}(f)$  is already contained in  $V^{(4)}$  so that  $T_\nu$  is already a smooth section on  $\widetilde{W}(f)$  by above proposition.

Now defined the inverse images  $D' = f^{-1}(V') \subset\subset D''' = f^{-1}(V''') \subset\subset D'' = f^{-1}(V'')$ . Note that by our assumption above, the relations such as  $\bar{D}' \subset f^{-1}(\bar{V}') \subset D'''$  hold. We may assume that  $D'$  here is the same as the one defined before. Since  $\bar{D}' \subset f^{-1}(\bar{V}') \subset f^{-1}(V''')$ , we may assume that for any  $h \in \widetilde{W}(f)$ , the image of  $h(f^{-1}(\bar{V}')) \subset V'''$ . Indeed since  $f(f^{-1}(\bar{V}'))$  is compact and contained in  $V'''$ , there exists a small  $\epsilon$ -neighborhood of  $f(f^{-1}(\bar{V}'))$  contained inside  $V'''$ . Hence for  $h \in \widetilde{W}(f)$  with  $\|h - f\| < \epsilon'$  with  $\epsilon'$  small enough, the image  $h(f^{-1}(\bar{V}')) \subset V'''$ . We will assume that any element  $h$  in  $\widetilde{W}(f)$  already has this property.

Then the non-empty compact set  $f(S^2 - D''')$  has no intersection with  $\bar{V}'$  since otherwise, there exists  $x \in S^2 - D'''$  such that  $f(x) \in \bar{V}'$  so that  $x \in f^{-1}(\bar{V}') \subset D'''$  contradiction with  $x \in S^2 - D'''$ . By the argument before we may assume that for all  $h \in \widetilde{W}(f)$ , we still have  $\bar{V}' \cap h(S^2 - D''') = \varnothing$ . In other words, for all  $h \in \widetilde{W}(f)$ , the inverse images  $h^{-1}(\bar{V}') \subset D'''$ .

Then by replace  $D$  and  $V$  by  $D''$  and  $V''$ , the proposition implies that  $T_\nu$  can be considered as a smooth section  $\widetilde{W}(f; D'') \rightarrow \widetilde{\mathcal{T}}_{k-1,p}(f; D'')$ , hence a smooth map  $T_\nu : \widetilde{W}(f; D'') \rightarrow L_{k-1}^p(D'', f_{D''}^*TV'') = (\widetilde{\mathcal{T}}_{k-1,p}(f; D''))_f$ . Let  $P_\beta : \widetilde{W}(f) \rightarrow L_{k-1}^p(S^2, \mathbf{R})$  defined by the pull backs of  $\beta$ ,  $P_\beta(h) = \beta \circ h$ . We will show in next lemma that  $P_\beta$  is of class  $C^\infty$ . Denote the multiplication map by  $m : L_{k-1}^p(S^2, \mathbf{R}) \times L_{k-1}^p(D'', f_{D''}^*TV'') \rightarrow L_{k-1}^p(D'', f_{D''}^*TV'')$  given by  $m(a, \xi) = R(a) \cdot \xi$ . Here  $R(a)$  is the restriction of  $a$  to  $D'$ . Then  $m$  is smooth. Now consider

$$F = m \circ (P_\beta, T_\nu) \circ (Id, Res) : \widetilde{W}(f) \rightarrow \widetilde{W}(f) \times \widetilde{W}(f; D'')$$

$$\rightarrow L_{k-1}^p(S^2, \mathbf{R}) \times L_{k-1}^p(D'', f_{D''}^* TV'') \rightarrow L_{k-1}^p(D'', f_{D''}^* TV'').$$

Here  $Res : \widetilde{W}(f) \rightarrow \widetilde{W}(f; D'')$  is the restriction map. Then it is smooth. It is easy to check that  $F(h) = (\beta \circ h) \cdot T_\nu(h)$  with obvious interpretations of the corresponding domains and ranges. Hence  $F$  is essentially equal to  $E_\beta(T_\nu) : \widetilde{W}(f) \rightarrow L_{k-1}^p(S^2, f^* TM)$  except the range of the map is  $L_{k-1}^p(D'', f_{D''}^* TV'')$ . However since the support of  $\beta$  is contained in  $V' \subset\subset V'''$ , the support of  $\beta \circ h$  is contained in the compact set  $h^{-1}(\bar{V}') \subset D'''$ . Now let  $\gamma$  be a cut-off function defined on  $S^2$  with support contained in  $D''$  and  $\gamma = 1$  on  $D'''$ . Then it induces a map  $E_\gamma : L_{k-1}^p(D'', f_{D''}^* TM) \rightarrow L_{k-1}^p(S^2, f^* TM)$  by multiplying with  $\gamma$ . Clearly  $E_\gamma$  is linear and continuous, and hence smooth. Now  $E_\beta(T_\nu) : \widetilde{W}(f) \rightarrow L_{k-1}^p(S^2, f^* TM)$  is equal to  $E_\gamma \circ F$ , hence smooth as well. □

**Lemma 2.1** *Let  $\beta : M \rightarrow \mathbf{R}$  be a smooth function. Then the map  $P = P_\beta : \widetilde{W}(f) \rightarrow L_k^p(S^2, \mathbf{R})$  defined by  $P(h) = \beta \circ h$  is of class  $C^\infty$ .*

**Proof:**

The proof is a simple application the theorem on smoothness of section functor in [P]. The two relevant finite dimensional bundles are  $p_1 : B_1 \subset E_1 = f^* TM \rightarrow S^2$  and the trivial bundle  $E_2 = S^2 \times \mathbf{R}^1 \rightarrow S^2$ . The bundle map  $p = p_\beta : B_1 \rightarrow E_2$  is defined by  $p(\xi) = (x, \beta(\exp_{f(x)} \xi))$  for  $\xi \in B_1$  with  $p_1(\xi) = x \in S^2$ .

Then by the theorem on section functor,  $\Gamma_{k,p}(p_\beta) : \Gamma_{k,p}(B_1) = \widetilde{W}(f) \rightarrow \Gamma_{k,p}(E_2) \simeq L_k^p(S^2, \mathbf{R})$  is of class  $C^\infty$ . It is easy to check that  $P_\beta = \Gamma_{k,p}(p_\beta)$ , hence of class  $C^\infty$  as well. □

**Corollary 2.2** *Let  $\beta : M \rightarrow \mathbf{R}^m$  be a smooth function. Then the map  $P = P_\beta : \widetilde{W}(f) \rightarrow L_k^p(S^2, \mathbf{R}^m)$  defined by  $P(h) = \beta \circ h$  is of class  $C^\infty$ .*

In particular, let  $X$  be a smooth vector field with support in a local chart  $V''$  of  $M$ . Assume that  $\dim(M) = m$  and  $\mathbf{t} = \{t_1, \cdot, t_m\}$  is a smooth local frame of  $TM$  on  $U$ . Then  $X = \sum_\nu X^\nu t_\nu$  with  $X^\nu : M \rightarrow \mathbf{R}$  supported in  $V''$ . Then  $\mathbf{P} = (P_{X^1}, \dots, P_{X^m}) : \widetilde{W}(f) \rightarrow L_k^p(S^2, \mathbf{R}^m)$  is of class  $C^\infty$ . Now take  $X = \beta t_\nu$  before. Then the map  $\mathbf{P}$  here is just the corresponding map for the

section  $E_\beta(T)$  in the previous theorem. This proved the section  $E_\beta(T)$  itself is smooth except that we are not using the standard local trivialization.

In the rest of this section, we will use several basic results whose proofs are much easier for the space  $L_k^p(S^2, \mathbf{R}^m)$  than for the general mapping space  $\mathcal{M}_{k,p}$ . The key step to reduce the proofs of theses results to the case  $L_k^p(S^2, \mathbf{R}^m)$  is the following proposition.

**Proposition 2.3** *Let  $\iota : M \rightarrow \mathbf{R}^m$  be an isometric embedding. Then the induced map  $\iota_* : \mathcal{M}_{k,p} \rightarrow L_k^p(S^2, \mathbf{R}^m)$  is a closed (and splitting)  $C^\infty$  embedding.*

**Proof:**

Unless it is a Hilbert space, for a general  $L_k^p$ -space, a close subspace may not have a complement. On the other hand, the usual definition of a closed submanifold of a Banach manifold requires the local splitting property so that most of the usual properties of submanifolds in finite dimensional case can be established accordingly for the infinite dimensional case. For our case here, the following implies the local splitting of the embedding.

For any smooth  $f : S^2 \rightarrow M$ ,

$$L_k^p(S^2, \mathbf{R}^m) \simeq L_k^p(S^2, f^*(T\mathbf{R}^m)) = L_k^p(S^2, f^*(TM)) \oplus L_k^p(S^2, f^*(N_M)).$$

Here  $N_M$  is the normal bundle of  $M$  in  $\mathbf{R}^m$  defined by  $(N_M)_m = \{v \in T_m\mathbf{R}^m \mid \langle v, T_m M \rangle = 0\}$  for any  $m \in M$  so that  $T_m\mathbf{R}^m = T_m M \oplus (N_M)_m$ .

The rest of the proof is a routine verification. □

In Section 4 we will give a general construction that implies the closeness of the embedding in above proposition.

The reparametrization group  $G = \mathbf{SL}(2, \mathbf{C})$  acts continuously on  $\mathcal{Map}$ . The following are proved in [L ?] :

(I) The action of  $G$  on  $\mathcal{Map} = \mathcal{Map}_{k,p}$  is  $G$ -Hausdorff so that the quotient space  $\mathcal{Map}/G$  is always Hausdorff.

(II) A  $L_k^p$ -map  $f \in \mathcal{Map}$  is said to be weakly stable with respect to  $G$  if its stabilizer  $\Gamma_f$  is compact. Let  $\tilde{\mathcal{B}}^w$  be the collection of all weakly stable maps in  $\mathcal{Map}$ . Then any element other than the constant ones in  $\mathcal{Map}$  is always weakly stable. Moreover the  $G$ -action on  $\tilde{\mathcal{B}}^w$  is proper so that for any  $f \in \tilde{\mathcal{B}}^w$  there is an open neighborhood  $U$  of  $f$  and a compact subset  $K$  of  $G$  such that for any  $g \in G \setminus K$  and  $h \in U$ ,  $g \cdot h \notin U$ .



(III) A weakly stable  $L_k^p$ -map is said to be stable if its stabilizer is finite. Let  $\tilde{\mathcal{B}} =: \tilde{\mathcal{B}}_{k,p}$  of the collection of the stable  $L_k^p$ -maps. Then for any  $f \in \tilde{\mathcal{B}}$ , there is a local slice  $S_f$  that is transversal to the all  $G$ -orbit  $\mathcal{O}_h$  at finitely many (uniformly bounded) points for  $h$  sufficiently close to  $f$ .

To simplifying our presentation, we assume that there is at least one point at which  $f$  is a local embedding of class  $m_0 \geq 1$ . This condition implies that  $f$  not only is stable but also has a local slice  $S_f$  constructed below using evaluation maps.

- Local slices  $S_f$  by evaluation map.

Recall the definition of the local slice  $S_f$  by evaluation map for  $f \in \tilde{\mathcal{B}}_{k,p}$  as follows.

It is sufficient to assume that  $f$  is of class  $C^\infty$ . Let  $\hat{W} = \hat{W}_\epsilon$  be the  $\epsilon$ -ball of  $L_k^p(S^2, f^*(TM))$ . Then we define  $\tilde{W}_f =: \text{Exp}_f \hat{W}$ . Here the exponential map is taken with respect to an  $f$  dependent metric specified below. Since there is at least one point, and hence any point in a neighborhood of that point, where  $f$  is a local embedding, we can fix three of such points as three standard marked points  $\mathbf{x} = \{x_1, x_2, x_3\} = \{0, 1, \infty\}$  on  $S^2$ .

Assume that the metric used to define  $\text{Exp}_f$  depending on  $f$  in the sense that it is flat near  $f(x_i) \in M, i = 1, 2, 3$ , such that an Euclidean neighborhood  $U_{f(x_i)}$  of  $f(x_i)$  is identified with a small ball  $B_{f(x_i)} \subset T_{x_i}M$  by  $\text{exp}_{f(x_i)} : B_{f(x_i)} \rightarrow U_{f(x_i)}$ . By this identification  $U_{f(x_i)}$  is decomposed as  $U_{f(x_i)} = V(f(x_i)) \oplus H(f(x_i))$  with two flat summands. Here  $V(f(x_i))$  is the local image of  $f$  near  $x_i$  and  $H(f(x_i))$  is the local flat hypersurface of codimension 2 transversal to  $f(x_i)$  at its origin. To define the local slices, we use the following lemma in [L].

**Proposition 2.4** *The 3-fold evaluation map at  $\mathbf{x}$ ,  $ev_{\mathbf{x}} : \tilde{W}_f \rightarrow M^3$  defined by  $ev_{\mathbf{x}}(h) = (h(x_1), h(x_2), h(x_3))$  is a smooth submersion.*

**Proof:**

It is sufficient to look at the case for the evaluation map  $ev_x$  with  $x$  being one of the  $x_i, i = 1, 2, 3$ .

We use the local charts  $\text{Exp}_f : \hat{W} \subset L_k^p(S^2, f^*(TM)) \rightarrow \tilde{W}_f$  and  $\text{exp}_{f(x)} : T_{f(x)}M \rightarrow M$ . Then under these coordinate charts,  $ev_x$  is given by

$$\hat{ev}_x : \xi \rightarrow \text{Exp}_f \xi \rightarrow \text{exp}_{f(x)}(\xi(x)) \rightarrow \xi(x).$$

In other words,  $\hat{ev}_x$  is just the restriction to  $\hat{W}$  of the evaluation map  $\hat{ev}_x : L_k^p(S^2, f^*(TM)) \rightarrow T_{f(x)}M$  given by  $\xi \rightarrow \xi(x)$ , which is linear and continuous, hence smooth.

□

Denote the three local flat hypersurfaces  $H(f(x_i)), i = 1, 2, 3$  together by  $\mathbf{H}$  and the corresponding flat local images  $V(f(x_i)), i = 1, 2, 3$  by  $\mathbf{V} = \mathbf{V}(f(\mathbf{x}))$  with coordinates  $\mathbf{v}$ . Let  $\pi_{\mathbf{V}} = \bigoplus_{i=1}^3 \pi_{V(f(x_i))}$  and  $\pi_{V_i(f)} : U_{f(x_i)} = V(f(x_i)) \oplus H(f(x_i)) \rightarrow V(f(x_i))$  be the projection map.

For  $\epsilon$  small enough, we define  $S_f = (ev_{\mathbf{x}})^{-1}(\mathbf{H}) = (\pi_{\mathbf{V}} \circ ev_{\mathbf{x}})^{-1}(0)$ . Then  $S_f$  is a  $C^\infty$  submanifold of codimension six in  $\widetilde{W}(f)$ , which is also denoted by  $\widetilde{W}(f; \mathbf{H})$ .

Let  $\hat{S}_f$  be the "lifting" of  $S_f$  in  $L_k^p(S^2, f^*(TM))$  so that  $S_f = Exp_f(\hat{S}_f)$ . Then  $\hat{S}_f$  is not an open ball in a linear subspace of  $L_k^p(S^2, f^*(TM))$  in general if the map  $Exp_f$  is defined by the fixed  $g_J$ -metric on  $M$ . However, with respect to the  $f$ -dependent metric above,  $H(f(x_i))$  is a geodesic submanifold so that for any  $\xi \in \hat{W} \subset L_k^p(S^2, f^*(TM))$ ,  $h = Exp_f \xi$  is in  $S_f$  if and only if  $\xi(x_i) \in T_{f(x_i)} H_i, i = 1, 2, 3$ . This implies that  $\hat{S}_f$  is the  $\epsilon$ -ball of the Banach space  $L_k^p(S^2, f^*(TM); T_{f(\mathbf{x})} \mathbf{H})$ . Here  $L_k^p(S^2, f^*(TM); T_{f(\mathbf{x})} \mathbf{H})$  is the linear subspace consisting of  $\xi \in L_k^p(S^2, f^*(TM))$  such that  $\xi(x_i)$  is in the tangent space  $T_{f(x_i)} H_i, i = 1, 2, 3$ .

To define the coordinate transformations between local slices, we need the following proposition proved in [L ] and [L ?].

**Proposition 2.5** *The composition of the action map with the 3-fold evaluation map,  $ev \circ \Psi_{\tilde{\mathcal{B}}} : G \times \tilde{\mathcal{B}} \rightarrow M^3$ , given by  $(g, h) = (h \circ g(x_1), h \circ g(x_2), h \circ g(x_3))$ , is of class  $C^{m_0}$ .*

Now assume that  $f' : S^2 \rightarrow M$  a stable  $L_k^p$ -map that is equivalent to  $f$  in the sense that (i) there is a biholomorphic map  $\phi : S^2 \rightarrow S^2$  such that  $f' = f \circ \phi$ ; (ii)  $f'$  is also a local embedding at the standard three marked points  $\mathbf{x}$ . Note that (ii) implies that  $f$  is a local embedding at both marked point sets  $\mathbf{x}$  and  $\mathbf{y} =: \phi(\mathbf{x})$ . Under these conditions, let  $S_{f'}$  be the local slice in  $\widetilde{W}_{f'}$ , and  $\Psi_S : S_f \rightarrow S_{f'}$  be the coordinate transformation. The following is proved in [L].

**Proposition 2.6** *Assume that  $m_0 > 1$ . Then there is a  $C^{m_0}$ -smooth function  $T : S_f \rightarrow G$  with  $T(f) = \phi$  such that for any  $h \in S_f$ ,  $\Psi_S(h) = h \circ T(h)$ .*

**Proof:**

Consider the map  $F =: \pi_{\mathbf{H}_{f'}^\perp} \circ \text{ev}_{f'(\mathbf{x})} \circ \Psi_{\tilde{\mathcal{B}}} : S_f \times G \rightarrow M^3 \rightarrow \mathbf{H}_{f'}^\perp$ , where  $\Psi_{\tilde{\mathcal{B}}}$  is the action map restricted to  $S_f \times G$ . Here  $\mathbf{H}_{f'}^\perp$  is the summand in the local decomposition of a flat neighborhood  $U_{f'}$  of  $M^3$  near  $f'(\mathbf{x})$ ,  $U_{f'} = \mathbf{H}_{f'} \oplus \mathbf{H}_{f'}^\perp$ . Note that as before here the local flat metric used is  $f'$ -dependent.  $\pi_{\mathbf{H}_{f'}^\perp} : U_{f'} \subset M^3 \rightarrow \mathbf{H}_{f'}^\perp$  is the projection with respect to above local decomposition. Then for  $(h, g) \in S_f \times G$ ,  $h \circ g$  is in  $S_{f'}$  if and only if  $F(h, g) = 0$ . In particular, for any  $h \in S_f$ , there is a  $g = T(h) \in G$  such that coordinate transformation  $\Psi_S(h) = h \circ T(h) \in S_{f'}$  so that  $(h, T(h))$  solves the equation  $F(h, g) = 0$ . Now under the assumption  $m_0 > 1$ , the function  $F$  is at least of class  $C^{m_0}$  with  $F(f, \phi) = 0$ . Moreover by the construction the partial derivatives along  $G$ ,  $\partial_g F|_{(f, \phi)} : T_\phi G \rightarrow (T\mathbf{H}_{f'(0)}^\perp)_0 \simeq \mathbf{H}_{f'}^\perp$  is surjective. Then by implicit function theorem, there is an unique such  $T : S_f \rightarrow G$  of class  $C^{m_0}$  with the desired property.  $\square$

The above argument implies the following corollary.

**Corollary 2.3** *Given a local slice  $S_f \subset \widetilde{W}(f)$ , there is a  $C^{m_0}$ -smooth function  $T : \widetilde{W}(f) \rightarrow G$  such that for any  $h \in \widetilde{W}(f)$ ,  $h \circ T(h) \in S_f$ .*

Now let  $\widetilde{W}(f) = \cup_{g \in G_e} g(S_f)$  be the decomposition of  $\widetilde{W}(f)$  by the images of  $S_f$  under the local  $G$ -action. Here  $G_e$  is the corresponding local group. When  $f$  is smooth, we may identify the  $\mathcal{O}_f^{G_e}$  with  $G_e$  smoothly. Then the above corollary implies the following corollary.

**Corollary 2.4** *There is a  $C^{m_0}$ -smooth projection map  $T_f : \widetilde{W}(f) \rightarrow \mathcal{O}_f^{G_e} \simeq G_e$  such that for any  $g \cdot f \in \mathcal{O}_f^{G_e}$ , the fiber  $T_f^{-1}(g \cdot f) = T^{-1}(g)$  is  $g(S_f)$  for any  $g \in G_e$ .*

- Local slices  $S_f$  by using  $L^2$ -metric on  $T_f L_k^p(S^2, \mathbf{R}^m)$

Still assume that the center  $f$  is of class  $C^\infty$  so that the orbit  $\mathcal{O}_f$  is a smooth submanifold of  $\tilde{\mathcal{B}}_{k,p} \subset L_k^p(S^2, \mathbf{R}^m)$ . Here  $\tilde{\mathcal{B}}_{k,p}$  is considered as a closed submanifold of  $L_k^p(S^2, \mathbf{R}^m)$  induced from an embedding  $M \subset \mathbf{R}^m$ . Then the tangent space  $T_f(\mathcal{O}_f)$  is a 6-dimensional linear subspace of

$$T_f \tilde{\mathcal{B}}_{k,p} = L_k^p(S^2, f^* TM) \subset T_f L_k^p(S^2, \mathbf{R}^m) = \{f\} \times L_k^p(S^2, \mathbf{R}^m).$$

Now for any  $\xi = (f, \eta) \in \{f\} \times C^\infty(S^2, \mathbf{R}^m) \subset T_f L_k^p(S^2, \mathbf{R}^m) = \{f\} \times L_k^p(S^2, \mathbf{R}^m)$ , consider the function  $I_\xi : L_k^p(S^2, \mathbf{R}^m) \rightarrow \mathbf{R}$  defined by  $I_\xi(h) = \langle$

$\xi, h - f >_2 =: \int_{S^2} \langle \xi, h - f \rangle d\text{vol}_{S^2}$ . In other words,  $I_\xi(h)$  is defined by the  $L^2$ -inner product on  $T_f L_k^p(S^2, \mathbf{R}^m)$  of  $\xi$  and the displacement from  $f$  to  $h$ . Let  $F_\xi = I_\xi \circ \Psi : G \times L_k^p(S^2, \mathbf{R}^m) \rightarrow L_k^p(S^2, \mathbf{R}^m) \rightarrow \mathbf{R}$ , where  $\Psi : G \times L_k^p(S^2, \mathbf{R}^m) \rightarrow L_k^p(S^2, \mathbf{R}^m)$  is the action map. Then we have the following proposition.

**Proposition 2.7** *The function  $F_\xi : G \times L_k^p(S^2, \mathbf{R}^m) \rightarrow \mathbf{R}$  is of class  $C^\infty$ .*

**Proof:**

For  $(\phi, h) \in G \times L_k^p(S^2, \mathbf{R}^m)$ ,

$$\begin{aligned} F_\xi(\phi, h) &= \int_{S^2} \langle \xi, h \circ \phi - f \rangle d\text{vol}_{S^2} \\ &= \int_{S^2} \langle \xi, h \circ \phi \rangle d\text{vol}_{S^2} - \int_{S^2} \langle \xi, f \rangle d\text{vol}_{S^2}. \end{aligned}$$

Thus upto a constant

$$\begin{aligned} F_\xi(\phi, h) &= \int_{S^2} \langle \xi, h \circ \phi \rangle d\text{vol}_{S^2} = \int_{S^2} \langle \xi \circ \phi^{-1}, h \rangle \det^{-1}(\phi) d\text{vol}_{S^2} \\ &= \int_{S^2} \langle \det^{-1}(\phi) \cdot \xi \circ \phi^{-1}, h \rangle d\text{vol}_{S^2}. \end{aligned}$$

Clearly  $F_\xi$  is a smooth function of class  $C^\infty$ . Indeed, Since  $\xi$  is fixed and smooth, the function  $A_\xi : G \rightarrow C^\infty(S^2, \mathbf{R}^m) \subset L_k^p(S^2, \mathbf{R}^m)$  given by  $A_\xi(\phi) = \det^{-1}(\phi) \cdot \xi \circ \phi^{-1}$  is smooth of class  $C^\infty$ . This together with the smoothness of the  $L^2$ -paring  $\langle -, - \rangle_2$  on  $L_k^p(S^2, \mathbf{R}^m)$  implies that  $F_\xi = \langle -, - \rangle_2 \circ (A_\xi, Id)$  is of class  $C^\infty$ .  $\square$

Let  $\hat{\mathbf{e}} = \{\hat{e}^1, \dots, \hat{e}^1\}$  be a basis of the Lie algebra  $T_e G$ , and  $\mathbf{e}_f = \{e_f^1, \dots, e_f^6\}$  be the induced basis of  $T_f(\mathcal{O}_f)$  by the infinitesimal action of  $G$  at  $e$ . Consider  $I_{\mathbf{e}_f} = (I_{e_f^1}, \dots, I_{e_f^6}) : L_k^p(S^2, \mathbf{R}^m) \rightarrow \mathbf{R}^6$  and  $F_{\mathbf{e}_f} = (F_{e_f^1}, \dots, F_{e_f^6}) : G \times L_k^p(S^2, \mathbf{R}^m) \rightarrow \mathbf{R}^6$ . Then both of them are of class  $C^\infty$ . So are their restrictions to  $L_k^p(S^2, M)$  and  $G \times L_k^p(S^2, M)$  respectively.

Now consider  $I_{\mathbf{e}_f} : L_k^p(S^2, M) \rightarrow \mathbf{R}^6$ . Then (1)  $I_{\mathbf{e}_f}(f) = \mathbf{0}$ ; (2) The derivative at  $f$ ,  $D_f(I_{\mathbf{e}_f})$  is surjective since the restriction of  $D_f(I_{\mathbf{e}_f})$  to  $T_f(\mathcal{O}_f)$  is. This implies that  $S_f^{L_2} =: (I_{\mathbf{e}_f})^{-1}(\mathbf{0})$  is a  $C^\infty$  submanifold of  $\widetilde{W}(f)$  with codimension six. It follows from the construction that  $S_f^{L_2}$  is a local slice for the local  $G$ -action on  $\widetilde{W}(f)$ .

If there is no confusion we will still denote  $S_f^{L^2}$  by  $S_f$ .

Now consider  $F_{e_f} : G_e \times \widetilde{W}(f) \rightarrow \mathbf{R}^6$ . Then (1)  $F_{e_f}(e, f) = \mathbf{0}$ ; (2) the partial derivative along  $G$ -direction at  $(f, e)$  is surjective. Here  $G_e$  is the local "group" near the identity  $e$  (= a small neighborhood of  $e$  in  $G$ ). Then it follows from the implicit function theorem that there is a  $C^\infty$  function  $T : \widetilde{W}(f) \rightarrow G$  such that the submanifold  $(F_{e_f})^{-1}(\mathbf{0})$  has the form  $\{(T(h), h) | h \in \widetilde{W}(f)\}$ . In other words, for any  $h \in \widetilde{W}(f)$ ,  $I_{e_f}(h \circ T(h)) = F_{e_f}(T(h), h) = \mathbf{0}$ , or equivalently  $h \circ T(h) \in S_f$ . Moreover, for given  $h$  if  $h \circ \phi \in S_f$  for some  $\phi \in G_e$ , then  $\phi = T(h)$ . This proves the following proposition.

**Proposition 2.8** *Given a local slice  $S_f^{L^2} \subset \widetilde{W}(f)$ , there is a  $C^\infty$ -smooth function  $T : \widetilde{W}(f) \rightarrow G$  such that for any  $h \in \widetilde{W}(f)$ ,  $h \circ T(h) \in S_f$ .*

**Note :** There is an obvious more natural way to define the local slice  $S_f^{L^2} \subset \widetilde{W}(f)$  by setting  $S_f^{L^2} = \text{Exp}_f \hat{S}$  where  $\hat{S} \subset L_k^p(S^2, f^*TM)$  is the  $\epsilon$ -ball of the  $L^2$ -orthogonal complement to the tangent space  $T_f(\mathcal{O}_f^G) \subset L_k^p(S^2, f^*TM)$ . Then the above proposition is still true. However its proof is slightly harder than the one above. This is the reason that we use the above definition here. A proof of the corresponding proposition as above for this new slice will be given in the forthcoming paper on the  $C^1$ -smoothness of the equivariant extension in the first construction in [L].

**Corollary 2.5** *There is a  $C^\infty$ -smooth projection map  $T_f : \widetilde{W}(f) \rightarrow \mathcal{O}_f^{G_e} \simeq G_e$  such that for any  $g \cdot f \in \mathcal{O}_f^{G_e}$ , the fiber  $T_f^{-1}(g \cdot f) = T^{-1}(g)$  is  $g(S_f^{L^2})$  for any  $g \in G_e$ .*

Given  $f$  and  $f'$  in  $\mathcal{B}$ , assume that the intersection of the orbits of the two slices,  $\mathcal{O}_{S_f} \cap \mathcal{O}_{S_{f'}} \neq \varnothing$  so that the transformation between these two slices, still denoted by  $\Psi_S$  is defined on  $S_f \cap \Psi_S^{-1}(S_{f'})$ .

**Corollary 2.6** *Assume that  $k - 2/p > 0$ . Then there is a  $C^\infty$ -smooth function  $T : S_f \cap \Psi_S^{-1}(S_{f'}) \rightarrow G$  with  $T(f) = \phi$  such that for any  $h \in S_f \cap \Psi_S^{-1}(S_{f'})$ ,  $\Psi_S(h) = h \circ T(h)$ .*

### 3 Proof of the Main Theorem

In this section we prove the following two versions of the main theorem.

- First version of the main theorem

Now start with a point-section

$$\eta \in C^\infty(S^2, f^*TM) \subset (\widetilde{\mathcal{T}}_{k-1,p})_f = L_{k-1}^p(S^2, f^*TM)$$

with

$$\begin{aligned} \eta &= \sum_i \alpha_i \cdot \eta = \sum_{i,\nu} a_i^\nu \cdot t_{i\nu} \circ f \\ &= \sum_{i,\nu} a_i^\nu \cdot \beta_i \circ f \cdot t_{i\nu} \circ f = \sum_{i,\nu} a_i^\nu \cdot E_{\beta_i}(t_{i\nu}) \circ f. \end{aligned}$$

Here  $a_i^\nu \in C^\infty(S^2, \mathbf{R})$  supported in  $D_i$  such that  $\alpha_i \cdot \eta = \sum_\nu a_i^\nu \cdot t_{i\nu} \circ f$ .

Recall the assumption that the local  $G$ -orbit  $\mathcal{O}_{S_f}$  of the local slice  $S_f$  in  $\widetilde{W}(f)$  is  $\widetilde{W}(f)$  itself.

Denote by  $\eta_{\mathcal{O}_{S_f}}$  the local  $G_e$ -equivariant extension of  $\eta$  over  $\mathcal{O}_{S_f} = \widetilde{W}(f)$ . We require that the extension  $\eta_{\mathcal{O}_{S_f}}$  to be defined satisfies the property that

$$\eta_{\mathcal{O}_{S_f}} = \sum_i (\alpha_i \cdot \eta)_{\mathcal{O}_{S_f}} = \sum_{i,\nu} (a_i^\nu)_{\mathcal{O}_{S_f}} \cdot (E_{\beta_i}(t_{i\nu}))_{\mathcal{O}_{S_f}}.$$

Thus we only need to defined the equivariant extensions  $(a_i^\nu)_{\mathcal{O}_{S_f}}$  and  $(E_{\beta_i}(t_{i\nu}))_{\mathcal{O}_{S_f}}$ .

Since  $E_{\beta_i}(T_{i\nu})$  is already smooth and  $G$ -equivariant defined on  $\mathcal{O}_{S_f} = \widetilde{W}(f)$ ,  $(E_{\beta_i}(t_{i\nu}))_{\mathcal{O}_{S_f}}$  is simply defined to be  $E_{\beta_i}(T_{i\nu})$ .

To define  $(a_i^\nu)_{\mathcal{O}_{S_f}}$ , let  $\mathcal{R} \rightarrow \widetilde{W}(f)$  be the trivial bundle  $\widetilde{W}(f) \times L_{k-1}^p(S^2, \mathbf{R}) \rightarrow \widetilde{W}(f)$ . Then  $a_i^\nu \in C^\infty(S^2, \mathbf{R}) \subset L_{k-1}^p(S^2, \mathbf{R})$  gives rise a constant section, denoted by  $(a_i^\nu)_{\widetilde{W}(f)} : \widetilde{W}(f) \rightarrow \mathcal{R}$ . Let  $(a_i^\nu)_{S_f} : S_f \rightarrow \mathcal{R}|_{S_f}$  be the restriction of  $(a_i^\nu)_{\widetilde{W}(f)}$  and  $[(a_i^\nu)_{S_f}] : S_f \rightarrow L_{k-1}^p(S^2, \mathbf{R})$  be the corresponding map under the trivialization of  $\mathcal{R}|_{S_f}$ . Note that  $(a_i^\nu)_{S_f}$  is still a constant section. The  $G$ -action on  $\mathcal{R}$  is given by to be  $\phi \cdot (h, a) = (h \circ \phi, a \circ \phi)$  for  $\phi \in G_e$  and  $(h, a) \in \mathcal{R} = \widetilde{W}(f) \times L_{k-1}^p(S^2, \mathbf{R})$ .

Now assume that  $S_f$  is one of the two slices defined in Sec. 2. Then we have a  $C^\infty$  or  $C^{m_0}$ -smooth map  $T : \widetilde{W}(f) \rightarrow G$  such that for any  $h \in \widetilde{W}(f)$ ,  $T(h) \circ h \in S_f$ . The  $G$ -equivariant extension  $(a_i^\nu)_{\mathcal{O}_{S_f}}$  is defined to be

$$(a_i^\nu)_{\mathcal{O}_{S_f}}(h) = (T(h))^*((a_i^\nu)_{S_f}(T(h) \circ h)).$$

Let  $[(a_i^\nu)_{\mathcal{O}_{S_f}}] : \widetilde{W}(f) \rightarrow L_{k-1}^p(S^2, \mathbf{R})$  be the corresponding map under the trivialization of  $\mathcal{R}|_{S_f}$ . Then  $[(a_i^\nu)_{\mathcal{O}_{S_f}}](h) = (T(h))^*(a_i^\nu) = a_i^\nu \circ T(h)$ .

**Lemma 3.1** *Assume that  $T : \widetilde{W}(f) \rightarrow G$  is of class  $C^\infty$  (or  $C^{m_0}$ ). The map  $[(a_i^\nu)_{\mathcal{O}_{S_f}}]$  is of class  $C^\infty$  (or  $C^{m_0}$ ) so that the  $G$ -equivariant extension  $(a_i^\nu)_{\mathcal{O}_{S_f}}$  is a smooth section.*

**Proof:**

Note that the action map

$\Psi_{(-l)} : G \times L_{k+l}^p(S^2, \mathbf{R}) \rightarrow L_k^p(S^2, \mathbf{R})$  is of class  $C^l$ . This implies that for a fixed  $a \in C^\infty(S^2, \mathbf{R}) \subset L_{k+l}^p(S^2, \mathbf{R})$ , the map  $G \rightarrow L_k^p(S^2, \mathbf{R})$  given by  $\phi \rightarrow a \circ h$  is of class  $C^l$  for any  $l$ , hence of class  $C^\infty$ . Now for  $a \in C^\infty(S^2, \mathbf{R})$ ,  $[a_{\mathcal{O}_{S_f}}]$  is the composition of two smooth maps,  $h \rightarrow T(h)$  from  $\widetilde{W}(f)$  to  $G$  and  $\phi \rightarrow a \circ \phi$  from  $G$  to  $L_k^p(S^2, \mathbf{R})$ . Hence  $[a_{\mathcal{O}_{S_f}}]$  is smooth.  $\square$

This proves the first version of the main theorem.

**Theorem 3.1** *Given a smooth stable map  $f$  and a smooth section*

$\eta \in L_{k-1}^p(S^2, f^*TM)$ , *its  $G$ -equivariant extension  $\eta_{\mathcal{O}_{S_f}}$  above as a section of  $\widetilde{\mathcal{T}}_{k-1,p} \rightarrow \widetilde{W}(f)$  has the same degree of smoothness as  $T$  has. Hence it is either of class  $C^{m_0}$  or of class  $C^\infty$  accordingly.*

• Second version of the main theorem

Similarly, for a given point-section

$$\xi \in C^\infty(S^2, \wedge^{0,1}(f^*TM)) \subset (\widetilde{\mathcal{L}}_{k-1,p})_f = L_{k-1}^p(S^2, \wedge^{0,1}(f^*TM))$$

with

$$\begin{aligned} \xi &= \sum_i \alpha_i \cdot \xi = \sum_{i,\nu} \gamma_i^\nu \cdot t_{i\nu} \circ f \\ &= \sum_{i,\nu} \gamma_i^\nu \cdot \beta_i \circ f \cdot t_{i\nu} \circ f = \sum_{i,\nu} \gamma_i^\nu \cdot E_{\beta_i}(t_{i\nu}) \circ f. \end{aligned}$$

Here  $\gamma_i^\nu \in C^\infty(S^2, \wedge^{0,1})$  supported in  $D_i$  such that  $\alpha_i \cdot \xi = \sum \gamma_i^\nu \cdot t_{i\nu} \circ f$ .

As before, the  $G$ -equivariant extension

$$\xi_{\mathcal{O}_{S_f}} = \sum_i (\alpha_i \cdot \xi)_{\mathcal{O}_{S_f}} = \sum_{i,\nu} (\gamma_i^\nu)_{\mathcal{O}_{S_f}} \cdot (E_{\beta_i}(t_{i\nu}))_{\mathcal{O}_{S_f}}.$$

We only need to defined the equivariant extensions  $(\gamma_i^\nu)_{\mathcal{O}_{S_f}}$ .

Since the bundle  $\wedge^{0,1} =: \wedge_{S^2}^{0,1} \rightarrow S^2$  is not trivial, we introduce an intermediate step as follows.

Note that the group  $G$  can be considered as the parametrized moduli space  $\widetilde{\mathcal{M}}(G) =: G$  consisting of holomorphic maps  $g : S^2 \rightarrow M = S^2$  of

class  $A = [id] = 1 \in H_2(S^2)$ . Let  $\omega^{0,1} = \widetilde{\mathcal{M}}(G) \times C^\infty(S^2, \wedge^{0,1})$  be the trivial bundle over  $\widetilde{\mathcal{M}}(G)$ . Here the norm used on the fiber  $C^\infty(S^2, \wedge^{0,1})$  is the  $L_k^p$ -norm. The group acts on  $C^\infty(S^2, \wedge^{0,1})$  by pull-backs,  $(g, \gamma) \rightarrow g^*(\gamma)$ . Thus the natural action of  $G$  on  $\widetilde{\mathcal{M}}(G)$  lifts to an action on the bundle  $\omega^{0,1}$ . Given an  $\gamma \in C^\infty(S^2, \wedge^{0,1}) = C^\infty(S^2, \wedge^{0,1})_e$  as a section over the base point  $e \in G = \widetilde{\mathcal{M}}(G)$ , let  $\gamma_G : \widetilde{\mathcal{M}}(G) \rightarrow \omega^{0,1}$  be its  $G$ -equivariant extension and  $[\gamma_G] : \widetilde{\mathcal{M}}(G) \rightarrow C^\infty(S^2, \wedge^{0,1})$  be the corresponding map under the trivialization.

**Lemma 3.2** *For any  $\gamma \in C^\infty(S^2, \wedge^{0,1})$ , the map  $[\gamma_G]$  is of class  $C^\infty$ . Thus the  $G$ -equivariant extension  $\gamma_G$  of the point-section  $\gamma$  is of class  $C^\infty$ .*

**Proof:**

Let  $\wedge^{0,1} = \wedge_{U_1}^{0,1} \cup \wedge_{U_2}^{0,1} \rightarrow S^2 = U_1 \cup U_2$  be the usual trivialization of the bundle  $\wedge^{0,1}$  with respect to the standard covering of  $S^2$ . Using the partition on unit  $\beta_1 + \beta_2 = 1$  with respect to the cover,  $\gamma = \beta_1\gamma + \beta_2\gamma$ . Then  $\gamma = \beta_1\gamma + \beta_2\gamma = a_1t_1 + a_2t_2$ . Here  $t_i, i = 1, 2$  is a smooth complex frame of  $\wedge_{U_i}^{0,1}$  and  $a_i$  is a complex valued smooth function supported in  $U_i$ . Now apply the argument before for the smoothness of  $E_{\beta_i}(T_{i\nu})$  to this simpler case by replacing  $TM$  and  $t_{i\nu}$  by  $\wedge^{0,1}$  and  $t_i, i = 1, 2$ . Then we get the  $G$ -equivariant extension  $[\gamma_G] : G \rightarrow C^\infty(S^2, \wedge^{0,1}) \subset L_{k-1}^p$  is of class  $C^\infty$ .  $\square$

Now define the trivial bundle  $\Omega^{0,1}|_{\mathcal{O}_{S_f}} \rightarrow \mathcal{O}_{S_f}$  to be the pull-back of  $\omega^{0,1} = \widetilde{\mathcal{M}}(G) \times C^\infty(S^2, \wedge^{0,1}) \rightarrow \widetilde{\mathcal{M}}(G)$  by the smooth or  $C^{m_0}$ -smooth map  $p_{\mathcal{O}_{S_f}} : \mathcal{O}_{S_f} \rightarrow G = \widetilde{\mathcal{M}}(G)$ . Here  $p_{\mathcal{O}_{S_f}}$  is defined by  $p_{\mathcal{O}_{S_f}}(h) = T(h)$  for  $h \in \mathcal{O}_{S_f}$ . Thus for  $\gamma_i^\nu \in C^\infty(S^2, \wedge^{0,1})$ , the  $G$ -equivariant extension of  $(\gamma_i^\nu)_{S_f}, (\gamma_i^\nu)_{\mathcal{O}_{S_f}}$ , is the same as the pull-back  $p_{\mathcal{O}_{S_f}}^*((\gamma_i^\nu)_G)$  of the smooth  $G$ -equivariant section  $(\gamma_i^\nu)_G$ . Here  $(\gamma_i^\nu)_{S_f}$  is defined to be the pull-back  $p_{S_f}^*(\gamma_i^\nu)$ . Therefore  $(\gamma_i^\nu)_{\mathcal{O}_{S_f}}$  is a smooth or  $C^{m_0}$ -smooth section.

This proves the second version the main theorem.

**Theorem 3.2** *Given a smooth stable map  $f$  and a smooth section*

$\xi \in L_{k-1}^p(S^2, \wedge^{0,1}(f^*TM))$ , *its "geometric"  $G$ -equivariant extension  $\xi_{\mathcal{O}_{S_f}}$  above has the same degree of smoothness as  $T$  has. Hence it is either of class  $C_0^m$  or of class  $C^\infty$  accordingly.*



## 4 The mapping space $\mathcal{M}ap_{k,p}(S^2, M)$ as Euler class of the section $\Delta$

There are several basic results used in this paper whose proof are much easier for the Banach space  $L_k^p(S^2, \mathbf{R}^m)$  than for the space  $\mathcal{M}ap_{k,p}(S^2, M) = L_k^p(S^2, M)$  of  $L_k^p$ -maps from  $S^2$  to a compact symplectic manifold  $M$ . We will introduce a general construction that realizes  $\mathcal{M}ap_{k,p}(S^2, M)$  as the zero locus of a smooth section  $\Delta$  of the bundle  $\mathcal{N} \rightarrow \mathcal{T}$  where  $\mathcal{T}$  is an tubular neighborhood of  $\mathcal{M}ap_{k,p}(S^2, M)$  in  $L_k^p(S^2, \mathbf{R}^m)$ .

Let  $\phi : M \rightarrow \mathbf{R}^m$  be an isometric embedding and  $p_M : N_M \simeq (T\mathbf{R}^m)|_M / TM \rightarrow M$  be the normal bundle of  $M$  in  $\mathbf{R}^m$ . Denote the corresponding bundle of  $\epsilon$ -balls in  $N_M$  by  $p_{B_M} : B_M \rightarrow M$ . When  $\epsilon$  small enough, the "exponential" map  $exp_{B_M} : B_M \rightarrow \mathbf{R}^m$  maps the bundle  $B_M$  to the tubular neighborhood  $T_M$  of  $M$ . Let  $p_{N_{B_M}} : N_{B_M} \rightarrow B_M$  be the pull-back of the normal bundle  $p_{N_M} : N_M \rightarrow M$  by the map  $p_{B_M} : B_M \rightarrow M$ . Using the inverse of the identification map  $exp_{B_M} : B_M \rightarrow T_M$  to pull-back the bundle  $p_{N_{B_M}} : N_{B_M} \rightarrow B_M$ , we get the bundle  $p_{N_{T_M}} : N_{T_M} \rightarrow T_M$  over the tubular neighborhood  $T_M$ . The tautological section  $\delta_{B_M} : B_M \rightarrow N_{B_M}$  given by  $\delta_{B_M}(b) = (b, b)$  gives rise the corresponding smooth section  $\delta_{T_M} : T_M \rightarrow N_{T_M}$  that is transversal to the zero section such that  $\delta_{T_M}^{-1}(0) = M$ .

Now consider the open set  $L_k^p(S^2, T_M)$  of the Banach space  $L_k^p(S^2, \mathbf{R}^m)$ . Define the bundle  $\mathcal{N}_{k,p} \rightarrow L_k^p(S^2, T_M)$  by requiring that  $(\mathcal{N}_{k,p})_f = L_k^p(S^2, f^*(N_{T_M}))$  for any  $f \in L_k^p(S^2, T_M)$ . The usual process in the standard GW theory implies that  $\mathcal{N}_{k,p}$  is indeed a  $C^\infty$  smooth bundle. The section  $\delta_{T_M} : T_M \rightarrow N_{T_M}$  induces the corresponding smooth section  $\Delta : L_k^p(S^2, T_M) \rightarrow \mathcal{N}_{k,p}$ . Then by its definition  $f \in L_k^p(S^2, T_M)$  lies inside  $L_k^p(S^2, M)$  if and only if  $\Delta(f) = 0$ . In other words,  $L_k^p(S^2, M)$  is just the zero locus  $\Delta^{-1}(0)$ . One can show that the section  $\Delta$  is transversal to the zero section so that  $L_k^p(S^2, M)$  obtained this way is an closed and splitting submanifold of  $L_k^p(S^2, \mathbf{R}^m)$  (see [La] for the definition of splitting submanifold). Indeed, the exponential map  $exp_{B_M} : B_M \rightarrow T_M$  induces a corresponding exponential map  $Exp_{B_M} : B_{L_k^p(S^2, M)} \rightarrow \mathcal{T} \subset L_k^2(S^2, T_M)$  where  $B_{L_k^p(S^2, M)} \rightarrow L_k^p(S^2, M)$  is the  $\epsilon_1$  ball bundle of  $\mathcal{N}_{k,p}|_{L_k^p(S^2, M)} \rightarrow L_k^p(S^2, M)$  and  $\mathcal{T}$  is the corresponding open tubular neighborhood of  $L_k^p(S^2, M)$  in  $L_k^2(S^2, T_M)$ . In other words, the restriction  $\mathcal{N}_{k,p}|_{L_k^p(S^2, M)} \rightarrow L_k^p(S^2, M)$  is just the normal bundle of  $L_k^p(S^2, M)$  in  $L_k^p(S^2, \mathbf{R}^m)$  so that  $\Delta$  can be interpreted as the corresponding tautological section defined on the tubular neighborhood  $\mathcal{T}$  as the finite dimensional case

before. In particular  $\Delta$  is transversal to the zero section. Thus we have realized the infinite dimensional manifold  $L_k^p(S^2, M)$  as the "Euler class" of the section  $\Delta$ .

Using above discussion, we now give a different analytic set-up for GW-theory as follows.

Note that the metric on  $\mathbf{R}^m$  induces a smooth metric on the tangent bundle  $T(T_M) \simeq T_M \times \mathbf{R}^m \rightarrow T_M$ . Since  $\exp_{B_M} : B_M \simeq T_M$ ,  $T_M$  itself can be considered as an open set of a vector bundle so that  $T(T_M) \simeq T(B_M) = V_{B_M} \oplus H_{B_M}$ . Here the vertical bundle is the same as  $p_{N_{B_M}} : N_{B_M} \rightarrow B_M$  while the horizontal bundle  $H_{B_M}$  is the bundle orthogonal to  $V_{B_M}$ . Clearly the derivative of projection map  $p_{B_M} : B_M \rightarrow M$  identifies each fiber of the horizontal bundle  $H_{B_M}$  with the corresponding fiber of  $TM$  so that  $H_{B_M}$  is identified with the pull-back  $p_{B_M}^*(TM) \simeq H_{B_M}$ . Now using the identification  $T_M \simeq B_M$ , we rewrite the above decomposition as  $T(T_M) = V_{T_M} \oplus H_{T_M}$  with the projection map  $\Pi_H : T(T_M) \rightarrow H_{T_M}$ . Then the almost complex structure  $J$  and symplectic form  $\omega$  becomes the corresponding fiberwise complex structure  $J_H$  and symplectic form  $\omega_H$  on the bundle  $H_{T_M}$ .

Now the linear operator  $d : L_k^p(S^2, \mathbf{R}^m) \rightarrow L_{k-1}^p(S^2, \wedge^1(\mathbf{R}^m))$  is obvious a smooth map. It can be interpreted as a smooth section of the trivial bundle  $\tilde{\Omega}_{k-1,p} = L_k^p(S^2, \mathbf{R}^m) \times L_{k-1}^p(S^2, \wedge^1(f^*(T\mathbf{R}^m))) \rightarrow L_k^p(S^2, \mathbf{R}^m)$  for any given  $C^\infty$  "reference" map  $f : S^2 \rightarrow \mathbf{R}^m$ . In particular, we may take the center  $f$  to be a smooth map  $f : S^2 \rightarrow M$ . Then by taking the restriction to the open set  $L_k^p(S^2, T_M)$ , we get the corresponding smooth section  $d : L_k^p(S^2, T_M) \rightarrow \tilde{\Omega}_{k-1,p}$ . Let  $d_H = (\Pi_H)_* \circ d : L_k^p(S^2, T_M) \rightarrow \tilde{\Omega}_{k-1,p}$  be the composed smooth section. Here  $(\Pi_H)_* : \tilde{\mathcal{T}}_{k-1,p} \rightarrow \tilde{\mathcal{H}}_{k-1,p} \subset \tilde{\mathcal{T}}_{k-1,p}$  is the  $C^\infty$  bundle endomorphism of  $\tilde{\mathcal{T}}_{k-1,p}$  induced by  $\Pi_H$ ; and  $\tilde{\mathcal{T}}_{k-1,p} = L_k^p(S^2, T_M) \times L_{k-1}^p(S^2, f^*(T\mathbf{R}^m)) \rightarrow L_k^p(S^2, T_M)$  is the trivial bundle with the "usual" trivialization and  $\tilde{\mathcal{H}}_{k-1,p}$  is the corresponding subbundle. The almost complex structure  $J_H$  can be interpreted as an endomorphism of  $T(T_M)$  by requiring that  $J_H = 0$  on the vertical bundle. Then  $J_H$  induces a  $C^\infty$  bundle endomorphism  $(J_H)_*$  of  $\tilde{\mathcal{T}}_{k-1,p}$  and hence an endomorphism, still denoted by  $(J_H)_*$  of the bundle  $\tilde{\Omega}_{k-1,p} \rightarrow L_k^p(S^2, T_M)$ . Finally, the complex structure  $i_0$  on  $S^2$  can be interpreted as a  $C^\infty$ -endomorphism  $i_0^*$  of the bundle  $\tilde{\Omega}_{k-1,p} \rightarrow L_k^p(S^2, T_M)$ .

Now we defined the lifted  $\bar{\partial}_{J_H}$  section of the bundle  $\tilde{\mathcal{L}}_{k-1,p} \rightarrow L_k^p(S^2, T_M)$  by  $\bar{\partial}_{J_H}(h) = \Pi_H(dh + (J_H)_*(h) \cdot i_0^*(dh))$ . Then  $\bar{\partial}_{J_H}$  is a smooth section of class  $C^\infty$ . Note that the section  $\bar{\partial}_{J_H}$  is in fact a section of the smooth subbundle

$\widetilde{\Omega}_{H;k-1,p}^{0,1} \rightarrow L_k^p(S^2, T_M)$ . Thus in term of the discussion above, the moduli space of  $J$ -holomorphic maps,  $\widetilde{\mathcal{M}}_J = \{f : S^2 \rightarrow M \mid \bar{\partial}_J(f) = df + J(f) \cdot df \circ i_0 = 0\}$  is just the intersection  $\Delta^{-1}(0) \cap \bar{\partial}_{J_H}^{-1}(0)$ . One can show that the smooth section  $(\Delta, \bar{\partial}_{J_H}) : L_k^p(S^2, T_M) \rightarrow \mathcal{N}_{k,p} \times \widetilde{\Omega}_{H;k-1,p}^{0,1}$  is in fact Fredholm with the index of  $D_f(\Delta, \bar{\partial}_{J_H}^{-1})$  is the same as the index of  $D_f \bar{\partial}_J$ . Thus we have realized the moduli space  $\widetilde{\mathcal{M}}_J$  as a Fredholm intersection of two infinite dimensional cycles in the flat ambient space  $L_k^p(S^2, T_M)$ .

The whole package for the analytic foundation of GW and Floer type theories can be carried out in this setting and its generalizations. That will lead to simplifications in many cases as here we are primarily dealing with the open set  $\mathcal{T}$  in the flat space  $L_k^p(S^2, \mathbf{R}^m)$ .

As an example, the standard basis  $\mathbf{t} = \{t_1, \dots, t_m\}$  of  $\mathbf{R}^m$  can be considered as a global frame of the tangent bundle  $T\mathbf{R}^m$ . Thus it gives rise the smooth global sections  $T = \{T_\nu, \nu = 1, \dots, m\}$  of the "tangent" bundle  $T(\mathcal{T})_{k-1,p} \rightarrow \mathcal{T}_{k,p}$ . This will simplify considerably the discuss of this paper on the  $G$ -equivariant extension  $\xi_{\mathcal{O}_{S_f}}$  for a point-section  $\xi$  if we use this new setting. The details of the discussions in this section will be treated in a separate paper.

## 5 Smoothness of $L_k^p$ -norm and related results

In this section we give a simpler proof of the smoothness of the  $L_k^p$ -norm. The main results of this section were proved in [C] and reproduced in [L]. Independent proofs with different settings were given in [CLW].

**Theorem 5.1** *Assume that  $p = 2m$  is a positive even integer. Let  $N_k(\xi) = \sum_{i=0}^k \int_\Sigma |D^i \xi|^p dvol_\Sigma$ . Then  $N_k : L_k^p(\Sigma, \mathbf{R}) \rightarrow \mathbf{R}$  is of class  $C^\infty$ . Here  $\Sigma$  is an oriented compact Riemannian manifold.*

**Proof:**

Note that  $N_k = \sum_{i=0}^k N^{(i)}$  where  $N^{(i)} : L_k^p \rightarrow \mathbf{R}$  is given by  $N^{(i)}(\xi) = \int_\Sigma |D^i \xi|^p dvol_\Sigma$ . Then  $N^{(i)} = N_0 \circ D^i : L_k^p \rightarrow L_{k-i}^p \rightarrow \mathbf{R}$ . Here  $D^i : L_k^p \rightarrow L_{k-i}^p$  given by  $\xi \rightarrow D^i(\xi)$  is linear and continuous, hence smooth. Thus  $N_k$  is smooth if and only if  $N_0$  is.

Now consider the polarization of  $N_0$ .  $P : (L_k^p(\Sigma, \mathbf{R}))_1 \times \dots \times (L_k^p(\Sigma, \mathbf{R}))_p \rightarrow \mathbf{R}$  given by  $P(\xi_1, \dots, \xi_{2m}) = \int_\Sigma \langle \xi_1, \xi_2 \rangle \dots \langle \xi_{2m-1}, \xi_{2m} \rangle dvol_\Sigma$ . Here each  $(L_k^p(\Sigma, \mathbf{R}))_j$  is a copy of  $L_k^p(\Sigma, \mathbf{R})$ .

The key point is that  $P$  is well-defined and continuous. Indeed, by Holder inequality for  $1/p_1 + 1/p_2 + \cdots + 1/p_{2m} = 1/r$  with  $p_1 = p_2 = \cdots = p_{2m} = 2m = p$  and  $r = 1$ ,

$$\begin{aligned} |P(\xi_1, \cdots \xi_{2m})| &= \left| \int_{\Sigma} \langle \xi_1, \xi_2 \rangle \cdots \langle \xi_{2m-1}, \xi_{2m} \rangle d\text{vol}_{\Sigma} \right| \\ &\leq \int_{\Sigma} |\xi_1| \cdot |\xi_2| \cdots |\xi_{2m-1}| \cdot |\xi_{2m}| d\text{vol}_{\Sigma} \leq \|\xi_1\|_p \cdot \|\xi_2\|_p \cdots \|\xi_{2m}\|_p. \end{aligned}$$

It is well-known in the usual Banach calculus that any continuous multi-linear function like  $P$  above is of class  $C^\infty$  (see Lang's book [La]).

Now  $N_0 = P \circ \Delta_{2m} : L_k^p \rightarrow (L_k^p(\Sigma, \mathbf{R}))_1 \times \cdots \times (L_k^p(\Sigma, \mathbf{R}))_p \rightarrow \mathbf{R}$ , where  $\Delta_{2m} : L_k^p \rightarrow (L_k^p(\Sigma, \mathbf{R}))_1 \times \cdots \times (L_k^p(\Sigma, \mathbf{R}))_p$  is the diagonal map that is smooth. Hence  $N_0$  is smooth. □

**Corollary 5.1** *The  $l$ -th derivative of  $N_0$  is equal to zero for  $l > p + 1 = 2m + 1$ .*

**Proof:**

The  $(p + 1)$ -th derivative of the multi-linear map  $P$  above is equal to zero (see [La]). The diagonal map  $\Delta_{2m}$  is linear so that its first and second derivatives are a constant map and zero respectively. Since  $N_0 = P \circ \Delta_{2m}$ , the conclusion follows from successively applying the chain rule and product rule for differentiations. □

Now assume that  $\Sigma = S^2$ .

**Corollary 5.2** *Let  $\Psi : G \times L_k^p(\Sigma, \mathbf{R}) \rightarrow L_k^p(\Sigma, \mathbf{R})$  be the action map. Here  $G = \mathbf{PSL}(2, \mathbf{C})$  acting on  $\Sigma = S^2$  as the group of reparametrizations. Then  $F_k = N_k \circ \Psi : G \times L_k^p(\Sigma, \mathbf{R}) \rightarrow \mathbf{R}$  is of class  $C^\infty$ .*

**Proof:**

We only give the proof for  $k = 0, 1$ . The general case can be proved similarly with more complicated notations.

- The case of  $k = 0$ .

$$F_0(\phi, \xi) = \int_{\Sigma} |\xi \circ \phi(x)|^p d\text{vol}_{\Sigma}(x) = \int_{\Sigma} |\xi|^p \det^{-1}(\phi) d\text{vol}_{\Sigma}.$$

Consider the "polarization" of  $F_0, P_0 : G \times (L^p(\Sigma, \mathbf{R}))_1 \times \cdots (L^p(\Sigma, \mathbf{R}))_p \rightarrow \mathbf{R}$  given by  $P_0(\phi, \xi_1, \cdots \xi_{2m}) = \int_{\Sigma} \langle \xi_1, \xi_2 \rangle \cdots \langle \xi_{2m-1}, \xi_{2m} \rangle \det^{-1}(Jac_{\phi}) dvol_{\Sigma}$ . As before,  $P_0$  is well-defined. Indeed

$$\begin{aligned} |P_0(\phi, \xi_1, \cdots \xi_{2m})| &\leq \|\det^{-1}(Jac_{\phi})\|_{C^0} \int_{\Sigma} |\xi_1| \cdot |\xi_2| \cdots |\xi_{2m-1}| \cdot |\xi_{2m}| dvol_{\Sigma} \\ &\leq \|\det^{-1}(Jac_{\phi})\|_{C^0} \|\xi_1\|_p \cdot \|\xi_2\|_p \cdots \|\xi_{2m}\|_p. \end{aligned}$$

Note that  $P_0$  is a composition of two smooth maps  $G \times (L^p(\Sigma, \mathbf{R}))_1 \times \cdots (L^p(\Sigma, \mathbf{R}))_p \rightarrow (L^p(\Sigma, \mathbf{R}))_0 \times (L^p(\Sigma, \mathbf{R}))_1 \times \cdots (L^p(\Sigma, \mathbf{R}))_p \rightarrow \mathbf{R}$  given by

$$\begin{aligned} (\phi, \xi_1, \cdots \xi_{2m}) &\rightarrow (\det^{-1}(Jac_{\phi}), \xi_1, \cdots \xi_{2m}) \\ &\rightarrow \int_{\Sigma} \langle \xi_1, \xi_2 \rangle \cdots \langle \xi_{2m-1}, \xi_{2m} \rangle \det^{-1}(Jac_{\phi}) dvol_{\Sigma}. \end{aligned}$$

Hence  $P_0$  and  $F_0$  is of class  $C^{\infty}$ .

• The case of  $k = 1$ .

Note that  $F_1 = F_0 + F^{(1)}$ . Here  $F_1^{(1)}$  is defined by

$$\begin{aligned} F_1^{(1)}(\phi, \xi) &= \int_{\Sigma} |\nabla(\xi \circ \phi)|^p dvol_{\Sigma} = \int_{\Sigma} |(\nabla \xi) \circ \phi(x) \cdot Jac_{\phi}(x)|^p dvol_{\Sigma}(x) \\ &= \int_{\Sigma} |\nabla \xi|^p \cdot |Jac_{\phi} \circ \phi^{-1}|^p \det^{-1}(Jac_{\phi}) dvol_{\Sigma}. \end{aligned}$$

Consider the "polarization" of  $F^{(1)}, P^{(1)} : G \times (L_1^p(\Sigma, \mathbf{R}))_1 \times \cdots (L_1^p(\Sigma, \mathbf{R}))_p \rightarrow \mathbf{R}$  given by  $P(\phi, \xi_1, \cdots \xi_{2m}) = \int_{\Sigma} \langle \nabla \xi_1, \nabla \xi_2 \rangle \cdots \langle \nabla \xi_{2m-1}, \nabla \xi_{2m} \rangle \cdot \langle (Jac_{\phi} \circ \phi^{-1}), (Jac_{\phi} \circ \phi^{-1}) \rangle^m \cdot \det^{-1}(Jac_{\phi}) dvol_{\Sigma}$ . Then  $P^{(1)}$  is well-defined. As before, it is a composition of two smooth maps and hence smooth. This implies that  $F_1$  is smooth. □

## References

- [C] X. Chen, Note on Partition of Unit, *Electronic file, dated Feb. 8, 2013*.
- [CLW] B. Chen, A. Li and B. Wang, Virtual Neighborhood Technique for Pseudo-holomorphic Spheres, *Preprint* (2013), arXiv:1306.3276 [math.GT].

- [F1] A. Floer, The Unregularized Gradient Flow of the Symplectic Action ,  
*Commun. Pure. Appl. Math.* **41**, 775-813 (1988).
- [La] S. Lang, Differential Manifolds, *Springer-Verlag* 1972.
- [H] H. Hofer, A General Fredholm Theory and Applications, *Preprint* (2005),  
arXiv: math.0509366[math.SG].
- [L] G. Liu, Weakly Smoothness in GW Theory, *Preprint* (2013 )  
arXiv:1310.7209 [math.SG].
- [McW] D. McDuff and K. Wehrheim, Smooth Kuranishi Structures with triv-  
ial isotropy, *Preprint* 2012.
- [P] R. Palais, Foundations of Global Non-linear Analysis, *Benjamin, INC*  
1968.